For a ring $R$ and a communicative group $(G, \cdot)$, we define $R|G := \{ \sum (r, g) : g \in G, r \in R \}$, the tuple $(r, g)$ is an element of $R \times G$.

Note that the addition of elements in $R \times G$ follows $(r_1, g) + (r_2, g) = (r_1 + r_2, g)$ where $r_1, r_2 \in R, g \in G$, and the addition of elements in $G$ is undefined.

The multiplication of elements in $R \times G$ is defined as $(r_1, g_1) \cdot (r_2, g_2) = (r_1 r_2, g_1 g_2)$

And the multiplication of elements in $R|G$ is defined as

$$\sum_{g \in G} (r_g, g) \cdot \sum_{h \in G} (r_h, h) = \sum_{k \in G} \sum_{hg=k} (r_g r_h, k)$$

For convenience, we set $(0, g) = 0$ for any $g \in G$, write $(r, g)$ as $rg$ or $gr$ and $(r, 1_G)$ as $r$

Properties:

1. When the ring $R$ is a field, $R|G$ becomes a vector space with basis $\{(1, g) : g \in G\}$

2. If the scalar multiplication $\cdot : R \times R|G \rightarrow R|G$ is defined as $r \cdot (s, g) := (r, 1_G) \cdot (s, g)$ where $r \cdot (s, g)$ can also be written as $(s, g) \cdot r$, then every element in $R|G$ is a linear combination of element in $\{(1, g) : g \in G\}$ over ring $R$.

Tensor product

Let $V, W$ be vector spaces over field $\mathbb{F}$, Suppose $B_V$ and $B_W$ are basis for $V$ and $W$, then $V \otimes W := \text{span}_\mathbb{F}(B_V \times B_W)$, instead of $(v_i, w_j)$, we write elements of $B_V \times B_W$ as $v_i \otimes w_j$ and note that the addition for elements in $V \otimes W$ is not defined as $v \otimes w + s \otimes t = (v + s) \otimes (w + t)$.
Addition of elements in $W \otimes V$ have following bilinear relations:

\[ k(v \otimes w) + m(u \otimes w) = (kv + mu) \otimes w \] for $k, m \in \mathbb{F}, u, v \in V, w \in W$\ldots (1)

\[ k(v \otimes w) + m(v \otimes t) = v \otimes (kw + mt) \] for $k, m \in \mathbb{F}, v \in V, u, t \in W$\ldots (2)

2016-05-17

For linear maps $f_1 : A \rightarrow B$, $f_2 : C \rightarrow D$, $f_1 \otimes f_2 : A \otimes C \rightarrow B \otimes D$ is the linear map defined as

\[ f_1 \otimes f_2(\sum_i a_i \otimes c_i) = \sum_i f_1(a_i) \otimes f_2(c_i) \]

Algebra

An algebra over a field $\mathbb{F}$ is a vectorspace $A$ with two linear maps:

$\mu : A \otimes A \rightarrow A$(multiplication) and $\eta : \mathbb{F} \rightarrow A$(embedding)

\[ A \otimes A \otimes A \xrightarrow{\mu \otimes id} A \otimes A \] \[ A \otimes A \xrightarrow{\mu} A \] \[ F \otimes A \xrightarrow{\eta \otimes id} A \otimes A \] \[ A \otimes F \xleftarrow{id \otimes \eta} A \otimes A \]

such that $\mu \circ (id \otimes \mu) = \mu \circ (\mu \otimes id)$ and $\mu$ commute, where $c$ and $c^*$ are scalar multiplication.

For algebra $A, B$, $g : A \rightarrow B$ is an algebra morphism if the diagram

\[ A \otimes A \xrightarrow{\mu} A \] \[ B \otimes B \xrightarrow{\mu_B} B \]

$g \otimes g$ commute.

Coalgebra

A coalgebra is a vector space $C$ over field $\mathbb{F}$ with two linear maps, comultiplication $\Delta : C \rightarrow C \otimes C$ and counit $\epsilon : C \rightarrow \mathbb{F}$ such the following diagrams commute
Element $c$ of coalgebra $C$ is group-like if $\Delta(c) = c \otimes c$ and $\epsilon(c) = 1$

The set of all group-like elements in $C$ is $G(C)$.

Bialgebra

A vectorspace $B$ is a bialgebra if $B$ is an algebra and $B$ is a coalgebra, and either multiplication and unit is coalgebra morphism or comultiplication and counit is algebra morphism.

For algebra $A$ and coalgebra $C$, if $f, g : C \rightarrow A$ are linear maps, then $f \star g(c) = \mu((f \otimes g)(\Delta(c)))$

Let $H$ be a bialgebra, an endomorphism $f$ is called an antipode of $H$ if $id \star f = f \star id = \eta \circ \epsilon$

A Hopf algebra is a bialgebra with antipode.

2016-05-18

Group representation

A matrix representation of a group $G$ is a homomorphism $R : G \rightarrow GL_n(\mathbb{C})$

A representation is faithful if it is injective.

The character of a matrix representation $R$ is the function $\chi_R(g) = trace(R(g))$

$\chi_R(1)$ is the dimension of representation.

characters are constant on conjugacy classes.

For a vectorspace $V$, $GL(V)$ is the group of all automorphisms of $V$, a representation of a group $G$ on a vectorspace $V$ is a homomorphism $\rho : G \rightarrow GL(V)$
Irreducible representations Let $\rho$ be a representation of a finite group $G$ on vectorspace $V$, a vector $v \in V$ is $G$-invariant if $\rho_g(v) = v$ for all $g \in G$.

For any $v \in V$, by averaging over the group $G$ could generate a $G$-invariant element $v'$. Where $v' = \frac{1}{|G|} \sum_{g \in G} \rho_g(v)$

A subspace $W$ of $V$ is $G$-invariant if $W$ is $g$-invariant subspace for all $g \in G$

If $\rho$ is the representation of $G$ on $V$ such that $V$ has no proper $G$-invariant subspace. Then $\rho$ is irreducible representation.

$\rho$ is a representation of group $G$ on vectorspace $V$, if $W_1$ and $W_2$ are $G$-invariant subspaces of $V$, and $V = W_1 \oplus W_2$, then we say $\rho = \alpha \oplus \beta$ where $\alpha$ is the representation of $G$ on $W_1$ and $\beta$ is the representation of $G$ on $W_2$

A representation $\rho$ of $G$ on a Hermitian space $V$ is unitary if $\rho_g$ is unitary for all $g \in G$

If $W$ is a $G$-invariant subspace, then $W^\perp$ is also $G$-invariant.

Note. Suppose $R$ is the matrix representation of a finite group $G$ on $V$. $G$ is finite implies for any $g \in G$, $g^n = 1_G$ for some $n < \infty$. Therefore $|R_g| = 1$ for all $g \in G$. 