Problem 1. Perhaps the easiest kind of surface to define is the graph of a function. Let $U \subset \mathbb{R}^{d-1}$ be open and let $\phi \in C^1(U, \mathbb{R})$. The graph of $\phi$ is the set

$$S = \{(y, \phi(y)) | y \in U \} \subset \mathbb{R}^d.$$ 

We may think of $S$ also as a parameterized $d-1$ surface given by the map

$$\Phi(y) = (y, \phi(y)).$$

Show that

$$\int_{S} f(x) \, dV_{d-1}(x) = \int_{U} f(\Phi(y)) \left[ 1 + \sum_{j=1}^{d-1} \left( \frac{\partial \phi}{\partial y_j}(y) \right)^2 \right] \, dy$$

for any continuous function $f$ defined on a neighborhood of $S$.

(You may assume that $f \circ \Phi$ has compact support in $U$. Here $V_{d-1}$ is the $d-1$ dimensional volume measure on $S$. Recall that the $j$-dimensional volume measure on a parameterized $j$-surface $\Phi \in C^1(U, \mathbb{R}^d)$ with $U$ open in $\mathbb{R}^j$ was defined in class via the formula

$$\int_{\Phi(U)} f(x) \, dV_j(x) = \int_{U} f(\Phi(y)) \sqrt{\det d\Phi(y)^T} \, d\Phi(y) \, dy.$$ )
Problem 2. This problem concerns integration on spheres in $\mathbb{R}^d$. Given $r > 0$ and $x_0$ in $\mathbb{R}^d$, let

$$S_r(x_0) := \{x| |x - x_0| = r\}$$

denote the $d-1$-sphere of radius $r$ with center $x_0$. The unit $d-1$-sphere centered at $0$ is also denoted

$$S^{d-1} := S_1(0) = \{x \in \mathbb{R}^d| |x| = 1\}.$$

1. Let $B^{d-1} = \{y \in \mathbb{R}^{d-1}: |y| < 1\}$ and let $\Phi_\pm : B_{d-1} \to \mathbb{R}^d$ be the maps

$$\Phi_\pm(y) = \left(y, \pm \sqrt{1 - |y|^2}\right).$$

So $\Phi_+(B^{d-1}) = S^{d-1}_+$ and $\Phi_-(B^{d-1}) = S^{d-1}_-$ are the upper and lower hemispheres

$$S^{d-1}_+ := \{x \in S^{d-1}| x_d > 0\}, \quad S^{d-1}_- := \{x \in S^{d-1}| x_d < 0\}.$$

Show that $\Phi_+$ and $\Phi_-$ are $C^1$, one-to-one and that their derivatives have full rank at every point of $B^{d-1}$.

2. Prove that $S_{d-1}$ is an oriented $d-1$-surface. (Hint: the maps $\Phi_\pm(y)$ defined on the unit ball $B_{d-1}$ give coordinates for the upper and lower hemispheres. Use similar maps to define a complete coordinate atlas. You will need to modify the chart functions to get consistent orientation.)

3. Prove that

$$\int_{S^{d-1}} f(\omega) dV_{d-1}(\omega) = \int_{B^{d-1}} \left[f(\Phi_+(y)) + f(\Phi_-(y))\right] \frac{1}{\sqrt{1 - |y|^2}} dy$$

for any continuous function $f : S^{d-1} \to \mathbb{R}$. (Hint: first show this if $f$ has compact support in the upper or lower hemispheres. Now use an approximation argument to obtain the result for general $f$.)

4. Show that

$$\int_{S_r(x_0)} f(y)dV_{d-1}(y) = r^{d-1} \int_{S^{d-1}} f(x_0 + r\omega) dV_{d-1}(\omega)$$

and, in particular, that

$$V_{d-1}(S_r(x_0)) = r^{d-1} V_{d-1}(S^{d-1}).$$
**Problem 3.** Polar coordinates in \( \mathbb{R}^d \) are obtained writing a point \( x \in \mathbb{R}^d \setminus \{0\} \) as \( x = r \omega \) where \( r = |x| \) and \( \omega = \frac{x}{|x|} \in S_{d-1} \).

1. Let \( f \) be a continuous compactly supported function on the upper half space \( \mathbb{H}^d_+ = \{ x \in \mathbb{R}^d | x_d > 0 \} \). Prove that

\[
\int_{\mathbb{H}^d_+} f(x) \, dx = \int_0^\infty \int_{S_{d-1}^+} f(r \omega) \, dV_{d-1}(\omega) \, r^{d-1} \, dr.
\]

(Hint: let \( x = r \Phi_+(y) \) with \( \Phi_+ \) as in the previous problem. Now use the change of variables formula to write the integral as an integral over \( r \) and \( y \). Then rewrite the integral over \( y \) as an integral over the upper hemisphere \( S_{d-1}^+ \).)

2. Now prove that

\[
\int_{\mathbb{R}^d} f(x) \, dx = \int_0^\infty \int_{S_{d-1}} f(r \omega) \, dV_{d-1}(\omega) \, r^{d-1} \, dr
\]

for any continuous compactly supported function \( f \) on \( \mathbb{R}^d \). (Hint: you may want to assume first that the support of \( f \) is contained in \( \mathbb{R}^d \setminus \{0\} \) and proceed to the general case using a limit.)

3. Use polar coordinates to show that

\[
V_d(B_r(0)) = \frac{V_{d-1}(S_{d-1}^{d-1})}{d} r^d
\]

where \( B_r(0) = \{ x \in \mathbb{R}^d | |x| < r \} \).
Problem 4. The goal of this exercise is to compute the $d-1$ Volume of the $d-1$ sphere.

1. Start by finding the exact value of the integral

$$\int_{\mathbb{R}^d} e^{-|x|^2} dx.$$

(Hint: Show that $\int_{\mathbb{R}^d} e^{-|x|^2} dx = \left(\int_{\mathbb{R}} e^{-x^2} dx\right)^d$ and then use exercise 2 from the last homework.)

2. Now show that

$$\int_{\mathbb{R}^d} e^{-|x|^2} dx = V_{d-1} \left(S^{d-1}\right) \int_0^\infty r^{d-1} e^{-r^2} dr.$$

3. Put the results together to prove that

$$V_{d-1} \left(S^{d-1}\right) = \frac{2\pi^{d/2}}{\Gamma \left(\frac{d}{2}\right)}.$$  \hspace{1cm} (1)

4. Use eq. (1) to show that $\Gamma \left(\frac{1}{2}\right) = \pi^{1/2}$. (Hint: what is $V_0 \left(S^0\right)$?)

5. Derive explicit formulas for $V_{d-1} \left(S^{d-1}\right)$ in dimensions $d = 2, 3, 4, 5, \text{ and } 6.$
Problem 5. A $C^2$ function $f$ is called harmonic if $\Delta f = 0$, where

$$\Delta f = \nabla \cdot \nabla f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}.$$ 

Let $f$ be $C^2$ and harmonic in an open subset $U$ of $\mathbb{R}^n$. The goal of this exercise is to prove the mean value property for $f$

$$\frac{1}{V_d(B_r(x))} \int_{B_r(x)} f(y) \, dy = f(x) \tag{2}$$

for any $r$ sufficiently small that $B_r(x) \subset U$.

1. Show that

$$\frac{1}{V_d(B_r(x))} \int_{B_r(x)} f(y) \, dy = \frac{d}{r^d} \int_0^r A_x(\rho) \rho^{d-1} d\rho$$

where

$$A_x(r) = \frac{1}{V_{d-1}(S^{d-1})} \int_{S^{d-1}} f(x + r\omega) dV_{d-1}(\omega).$$

2. Prove that

$$\lim_{r \to 0} A_x(r) = f(x).$$

3. Show that

$$\frac{d}{dr} A_x(r) = \frac{1}{V_{d-1}(S^{d-1})} \int_{S^{d-1}} \nabla f(x + r\omega) \cdot \omega \, dV_{d-1}(\omega),$$

and use the divergence theorem (a corollary of Stoke’s theorem we proved in class) together with the fact that $f$ is harmonic to prove that $\frac{d}{dr} A_x(r) = 0$ provided $B_r(x) \subset U$.

4. Conclude that

$$A_x(r) = f(x)$$

provided $B_r(x) \subset U$ and use this to prove eq. (2).