We now turn to the study of hyper-surfaces in $\mathbb{R}^d$. A hyper-surface is something like a curve or a surface in $\mathbb{R}^3$ or their higher dimensional analogues. A differential form is the something we can integrate over a hyper-surface.

For example, a parameterized curve in $\mathbb{R}^3$ is a map $\gamma : [0, 1] \to \mathbb{R}^3$, we will call this a $C^1$ curve if $\gamma$ is $C^1$. Given such a curve with $\gamma([0, 1]) \subset U$ and a function $F : U \to \mathbb{R}^d$ we can form the integral of $\text{“}F(x) \cdot dx\text{”}$ along $\gamma$:

$$
\int_\gamma F(x) \cdot dx := \int_0^1 F(\gamma(t)) \cdot \gamma'(t)dt.
$$

We will call the expression $F(x) \cdot dx = \sum_{j=1}^d F_j(x) dx_j$ a differential one form. We will generalize this notion to make differential $j$-forms and integrate these over hypersurfaces of dimension $j$.

Hypersurfaces

**Definition 1.** A parameterized $j$-surface in $\mathbb{R}^d$ is a $C^1$ map $\Phi : V \to \mathbb{R}^d$ where $V \subset \mathbb{R}^j$ is open. If $\Phi(V) \subset U$ with $U \subset \mathbb{R}^d$ an open set, we say that $\Phi$ is a parameterized $j$-surface in $U$.

**Remark 2.** We will call $\Phi$ a $j$-surface for short. In principle we should define a $j$-surface as an equivalence class of parameterized $j$-surface where we make $\Phi_1 \sim \Phi_2$ if $\Phi_1 = \Phi_2 \circ T$ with $T$ a one-to-one $C^1$ map with $\det dT(x) \neq 0$ at all points of its domain. Think of a curve, say $\Phi_1(t) = (\cos t, \sin t) \in \mathbb{R}^2$ where $t \in [0, 2\pi]$. This is the closed unit circle in $\mathbb{R}^2$. As a parametrized curve this is distinct from $\Phi_2(t) = (\cos 2t, \sin 2t)$ defined on the domain $[0, \pi]$, but we see that the difference is simply one of reparameterizing the domain. We should keep this in mind, but for the sake of simplicity we will refer to parameterized $j$-surfaces simply as $j$-surfaces.

Differential forms

**Definition 3.** Let $U \subset \mathbb{R}^d$ be an open set and $0 \leq j \leq d$ an integer. A differential $j$-form is a function $\omega$ from $U$ to the space $\Lambda^j(\mathbb{R}^d)$ of alternating $j$-forms on $\mathbb{R}^d$. If $\omega \in C^\alpha(U, \Lambda^j(\mathbb{R}^d))$ with $\alpha$ a non-negative integer we say that $\omega$ is a $C^\alpha$ $j$-form or that $\omega$ is of class $C^\alpha$ (for $\alpha = 0$ we also say that $\omega$ is a continuous $j$-form).
Remark 4. Since \( \Lambda^0(\mathbb{R}^d) = \mathbb{R} \) by definition, a differential 0-form on \( U \) is just a real valued function on \( U \).

We will develop some special notation for \( j \)-forms in a bit. For the moment we will denote the value of \( \omega \) at a point \( \mathbf{x} \in U \) by \( \omega(\mathbf{x}) \). Note that for each \( \mathbf{x} \), \( \omega(\mathbf{x}) \) is an alternating \( j \)-form, which in particular is a function taking as an argument \( j \)-vectors in \( \mathbb{R}^d \). We will denote the action of \( \omega(\mathbf{x}) \) on the vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_j \) by

\[
\omega(\mathbf{x}) [\mathbf{v}_1, \ldots, \mathbf{v}_j].
\]

Remark 5. \( \omega \) is \( C^\alpha \) if and only if for each \( \mathbf{v}_1, \ldots, \mathbf{v}_j \in \mathbb{R}^d \) the map \( \mathbf{x} \mapsto \omega(\mathbf{x}) [\mathbf{v}_1, \ldots, \mathbf{v}_j] \) is \( C^\alpha \).

Lemma 6. Suppose \( \omega \) is a \( j \)-form on \( U \subset \mathbb{R}^d \) and \( \Phi : V \rightarrow \mathbb{R}^d \) is a parameterized \( k \)-surface. Then

\[
\omega_\Phi(\mathbf{x}) [\mathbf{v}_1, \ldots, \mathbf{v}_j] := \omega(\Phi(\mathbf{x})) [d\Phi(\mathbf{x}) \cdot \mathbf{v}_1, \ldots, d\Phi(\mathbf{x}) \cdot \mathbf{v}_j].
\]

is a \( j \)-form on \( V \), which is continuous if \( \omega \) is.

Proof. That \( \omega_\Phi \) is a differential \( j \)-form follows easily from the fact that \( \omega \) is. The continuity follows easily from the continuity of \( \Phi \) and \( d\Phi \).

Remark 7. The form \( \omega_\Phi \) is called the pullback of \( \omega \) under \( \Phi \).

Definition 8. Let \( \omega \in C(U, \Lambda^j(\mathbb{R}^d)) \) be a continuous \( j \)-form, let \( \Phi : V \rightarrow \mathbb{R}^d \) be a parameterized \( j \)-surface in \( U \) and let \( \mathbf{s}_1, \ldots, \mathbf{s}_j \) be the standard basis of \( \mathbb{R}^j \). Then the integral of \( \omega \) over \( \Phi \) is

\[
\int_\Phi \omega := \int_V \omega_\Phi(\mathbf{x}) [\mathbf{s}_1, \ldots, \mathbf{s}_j] d\mathbf{x} \quad \text{(1)}
\]

whenever the integral on the right hand side is defined.

Remark 9. 1) For the most part, it will suffice to consider \( \int_\Phi \omega \) only in case \( \omega_\Phi \) has compact support in \( V \). (Here the support of a \( j \)-form is the closure of the set on which it is not identically zero.) In that case the integral on the right hand side of (1) is well defined, since we may continuously extend \( \omega_\Phi \) to all of \( \mathbb{R}^j \) by taking it to be zero off of \( V \) and set

\[
\int_\mathbb{R}^j \omega_\Phi(\mathbf{x}) [\mathbf{s}_1, \ldots, \mathbf{s}_j] = \int_{\mathbb{R}^j} \omega_\Phi(\mathbf{x}) [\mathbf{s}_1, \ldots, \mathbf{s}_j] d\mathbf{x}.
\]

2) Sometimes it is useful to consider a \( j \)-surface defined on a \( j \)-cell \( I \) in \( \mathbb{R}^j \). If \( \Phi \) and \( d\Phi \) extend continuously to the boundary points of \( I \) then the integral on the right
hand side of (1) is well defined as an iterated integral. 3) For a $k$-form $\omega$ in $\mathbb{R}^k$ we will simply write
\[
\int_{I^k} \omega := \int_{I^k} \omega(x) [e_1, \ldots, e_k] \, dx
\]
with $e_1, \ldots, e_k$ the standard basis and $I^k$ a $k$-cell. This amounts to identifying the $k$-cell with the $k$-surface given by the identity map.

Why is (1) a good definition? The answer comes from considering change of variables. Indeed, suppose $\Phi_1$ and $\Phi_2$ are parameterized $j$ surfaces in $U$ with $\Phi_1 = \Phi_2 \circ T$ where $T$ is a one-to-one map with $\det dT(x) \neq 0$ everywhere. Then
\[
\begin{align*}
\omega_{\Phi_1}(x)[s_1, \ldots, s_j] &= \omega(\Phi_2(T(x)))[d\Phi_2(T(x)) \cdot dT(x) \cdot s_1, \ldots, d\Phi_2(T(x)) \cdot dT(x) \cdot s_j] \\
&= \omega_{\Phi_2}(T(x))[dT(x) \cdot s_1, \ldots, dT(x) \cdot s_j] \\
&= \det dT(x) \cdot \omega_{\Phi_2}(T(x))[s_1, \ldots, s_j],
\end{align*}
\]
by the definition of the derivative. If the domain $V_1$ of $\Phi_1$ is connected then we either have $\det dT(x) \cdot > 0$ or $\det dT(x) \cdot < 0$ for each $x \in V_1$. Thus, by the change of variables formula,
\[
\int_{\Phi_1} \omega = \int_{V_1} \omega_{\Phi_1} = \int_{V_1} \det dT(x) \cdot \omega_{\Phi_2}(T(x))[s_1, \ldots, s_j] = \pm \int_{V_2} \omega_{\Phi_2} = \pm \int_{\Phi_2} \omega,
\]
with the $\pm$ sign corresponding to the sign of $\det dT(x) \cdot$.

What does this $\pm$ sign mean? Remember that for the fundamental theorem of calculus in $d = 1$ it was useful to define $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$ for $a < b$, thus making the integral into an oriented integral that depends on the orientation of the interval. In the same way, a $j$-surface in $\mathbb{R}^d$ has two possible orientations and switching orientations results in a minus sign. You may have encountered this in vector calculus when integrating over 2 surfaces in $\mathbb{R}^3$ — there are two choices of unit normal vectors and the choice determines the sign of integrals like $\int_{\Sigma} F(x) \cdot \hat{n}(x) \, d\sigma(x)$.

**Wedge product and elementary forms**

**Definition 10.** Given an alternating $j$-form $\omega$ and an alternating $k$-form $\nu$ we define the **wedge product** of $\omega$ and $\nu$ to be the alternating $j+k$ from $\omega \wedge \nu$ by
\[
\omega \wedge \nu[v_1, \ldots, v_{j+k}] := \frac{1}{j!k!} \sum_{\sigma \in S_{j+k}} \text{sgn} \sigma \, \omega[v_{\sigma(1)}, \ldots, v_{\sigma(j)}] \nu[v_{\sigma(j+1)}, \ldots, v_{\sigma(j+k)}],
\]
where the sum runs over the set $S_{j+k}$ of all permutations of $\{1, \ldots, j+k\}$. For differential $j$-forms $\omega$ and $\nu$ in a domain $U$ we define the wedge product pointwise in $x \in U$: definition extends to differential

$$[\omega \wedge \nu](x) := (\omega(x)) \wedge (\nu(x)). \quad (2)$$

**Remark 11.** Note that given a $j$-form $\omega$ and a $j'$-form $\nu$ on an open set $U \subset \mathbb{R}^d$. Given a parameterized $k$-surface $\Phi : V \to U$ we have

$$[\omega \wedge \nu]_{\Phi} = \omega_{\Phi} \wedge \nu_{\Phi}.$$

**Remark 12.** The “wedge” product of a zero form $f$ and a $j$-form $\omega$ is the $j$-form $[f \wedge \omega](x) [v_1, \ldots, v_j] = f(x) \omega(x) [v_1, \ldots, v_j]$.

We will typically denote this product as just $f \omega$.

**Exercise 13.** Let $\omega$ be an alternating $j$-form, $\nu$ be an alternating $k$-form and $\alpha$ an alternating $l$-form. Show that $\omega \wedge \nu = (-1)^{kj} \nu \wedge \omega$ and that $(\omega \wedge \nu) \wedge \alpha = \omega \wedge (\nu \wedge \alpha)$.

**Exercise 14.** Let $\omega_1$ and $\omega_2$ be alternating $j$-forms and let $\nu$ be an alternating $k$-form. Show that $(\omega_1 + \omega_2) \wedge \nu = \omega_1 \wedge \nu + \omega_2 \wedge \nu$.

To proceed it is useful to introduce a notation for differential forms. First we define the *elementary* 1 forms $dx_1, \ldots, dx_d$ on $\mathbb{R}^d$ by

$$dx_j(v) = v_j,$$

so $dx_j(e_i) = 0$ if $i \neq j$ and $= 1$ if $i = j$. These forms make a a basis for $\Lambda^1(\mathbb{R}^d)$. Then for each $j$ we define the *elementary* $j$ forms to be the following

$$dx_{\alpha_1} \wedge dx_{\alpha_2} \wedge \cdots \wedge dx_{\alpha_j}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_j \in \{1, \ldots, d\}$ — this makes sense without parentheses by exercise 13. For instance

$$dx_{\alpha} \wedge dx_{\beta} [v_1, v_2] = v_{1,\alpha} v_{2,\beta} - v_{1,\beta} v_{2,\alpha}.$$

Note that if $\alpha_i = \alpha_{i'}$ for any $i \neq i'$ then the resulting form is zero and more generally that

$$dx_{\alpha_{\sigma(1)}} \wedge \cdots \wedge dx_{\alpha_{\sigma(j)}} = \text{sgn} \sigma dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_j},$$

for any permutation $\sigma$ of $\{1, \ldots, j\}$. Hence, up to sign, there are only $\binom{d}{j}$ elementary $j$ forms, given by

$$dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_j}$$

where $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_j \leq d$. 

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Let us introduce a compact notation for these elementary $j$-forms. Given $S \subset \{1, \ldots, d\}$ we can write its elements in increasing order $\alpha_1 < \cdots < \alpha_j$ where $\#S = j$. Let
\[ dx_S := dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_j}. \]
Note that given $S \subset \{1, \ldots, d\}$ of size $j$,
\[ dx_S \left( e_{\alpha_1}, \ldots, e_{\alpha_j} \right) = \begin{cases} 0 & S \neq \{\alpha_1, \ldots, \alpha_j\} \\ \text{sgn} \sigma & S = \{\alpha_1, \ldots, \alpha_j\} \end{cases} \]
with $\sigma$ is the unique permutation of $\{1, \ldots, j\}$ that puts $\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(j)}$ in increasing order.

**Theorem 15.** Let $\omega$ be a differential $j$-form on an open set $U \subset \mathbb{R}^d$ then to each subset $S \subset \{1, \ldots, d\}$ of size $j$ there is a function $f_S$ on $U$ such that
\[ \omega(x) = \sum_S f_S(x) \, dx_S. \]
Furthermore $\omega$ is $C^\alpha$ if and only if the functions $f_S \in C^\alpha$ for each $S$.

**Proof.** Given $S$ of size $j$ with $S = \{\alpha_1 < \alpha_2 < \cdots < \alpha_j\}$ let
\[ f_S(x) = \omega(x) \left[ e_{\alpha_1}, \ldots, e_{\alpha_j} \right]. \]
Clearly if $\omega$ is $C^\alpha$ then $f$ is also. Furthermore
\[ \omega(x) \left[ v_1, \ldots, v_j \right] = \sum_{\beta_1, \ldots, \beta_j=1}^d v_{\beta_1} ; \cdots ; v_{\beta_j} \omega(x) \left[ e_{\beta_1}, \ldots, e_{\beta_j} \right]. \]
Now for each $\beta_1, \ldots, \beta_j$ we have
\[ \omega(x) \left[ e_{\beta_1}, \ldots, e_{\beta_j} \right] = f_S(x) \, dx_S \left( e_{\beta_1}, \ldots, e_{\beta_j} \right) \]
where $S = \{\beta_1, \ldots, \beta_j\}$. Note also that
\[ f_{S'}(x) \, dx_{S'} \left( e_{\beta_1}, \ldots, e_{\beta_j} \right) = 0 \]
if $S' \neq \{\beta_1, \ldots, \beta_j\}$. Thus
\[ \omega(x) \left[ v_1, \ldots, v_j \right] = \sum_{\beta_1, \ldots, \beta_j=1}^d v_{\beta_1} ; \cdots ; v_{\beta_j} f_S(x) \, dx_S \left( e_{\beta_1}, \ldots, e_{\beta_j} \right) \]
\[ = \sum_S \sum_{\beta_1, \ldots, \beta_j=1}^d v_{\beta_1} ; \cdots ; v_{\beta_j} f_S(x) \, dx_S \left( e_{\beta_1}, \ldots, e_{\beta_j} \right) \]
\[ = \sum_S f_S(x) \, dx_S \left( v_1, \ldots, v_j \right) \]
as claimed.
Since $dx_S(v_1,\ldots,v_j)$ is constant as a function of $x$, it is $C^\alpha$ so that if $f_S$ is $C^\alpha$ for each $S$ then $\omega$ is too.

**Definition 16.** Let $\Phi : V \to \mathbb{R}^d$ be a parameterized $k$-surface in $\mathbb{R}^d$. Let points in $V$ be denoted $y$ and points in $\Phi(V)$ be denoted $x$, so $x = \Phi(y)$. Given $\alpha_1,\ldots,\alpha_j \in \{1,\ldots,d\}$ and $\beta_1,\ldots,\beta_j \in \{1,\ldots,k\}$ define the Jacobian

$$\frac{\partial(x_{\alpha_1},\ldots,x_{\alpha_j})}{\partial(y_{\beta_1},\ldots,y_{\beta_j})} := \det \left( \frac{\partial \Phi_{\alpha_i}(y)}{\partial y_{\beta_{i'}}} \right)_{i,i'=1}^j.$$

Given $S = \{\alpha_1 < \ldots < \alpha_j\} \subset \{1,\ldots,d\}$ and $S' = \{\beta_1 < \ldots < \beta_j\}$ let $\frac{\partial x_S}{\partial y_{S'}}$ denote the corresponding Jacobian with $\alpha$’s and $\beta$’s in increasing order.

**Proposition 17.** Let $\Phi$ be a parameterized $k$-surface in $\mathbb{R}^d$ and $\omega = \sum_S f_S dx_S$ a $j$-form defined on a neighborhood of $\Phi$. Then

$$\omega_{\Phi}(y) = \sum_{S'} \left( \sum_S f_S(\Phi(y)) \frac{\partial x_S}{\partial y_{S'}} \right) dy_{S'}.$$

**Proof.** Let $s_1,\ldots,s_k$ denote the standard basis of $\mathbb{R}^k$ and $e_1,\ldots,e_d$ the standard basis of $\mathbb{R}^d$. Then

$$\Phi'(y)s_\beta = \sum_{\alpha=1}^d \frac{\partial \Phi_{\alpha}(x)}{\partial y_\beta} e_\alpha.$$

Thus given $S' = \{\beta_1 < \ldots < \beta_j\} \subset \{1,\ldots,k\}$,

$$\omega_{\Phi}(y)[s_{\beta_1},\ldots,s_{\beta_j}] = \omega(\Phi(y))[\Phi'(y)s_{\beta_1},\ldots,\Phi'(y)s_{\beta_j}]$$

$$= \sum_{\alpha_1,\ldots,\alpha_j=1}^d \frac{\partial \Phi_{\alpha_1}}{\partial y_{\beta_1}} \ldots \frac{\partial \Phi_{\alpha_j}}{\partial y_{\beta_j}} \omega(\Phi(y))[e_{\alpha_1},\ldots,e_{\alpha_j}]$$

$$= \sum_S f_S(x) \sum_{\sigma} \text{sgn} \sigma \frac{\partial \Phi_{\alpha_{\sigma(1)}}}{\partial y_{\beta_1}} \ldots \frac{\partial \Phi_{\alpha_{\sigma(j)}}}{\partial y_{\beta_j}}$$

$$= \sum_S f_S(x) \frac{\partial x_S}{\partial y_{S'}},$$

where in the second to last line $S = \{\alpha_1,\ldots,\alpha_j\}$. Together with the previous theorem this proves the result.

A particular case of this result is when $k = j$, so for $\omega = \sum_S f_S dx_S$ we get the explicit expression

$$\int_{\Phi} \omega = \sum_S \int_V f_S(\Phi(y)) \frac{\partial x_S}{\partial y_{\{1,\ldots,j\}}} dy.$$
Exterior derivative

Let $U$ be an open set in $\mathbb{R}^d$. If $f \in C^1(U)$ we define

$$df := \sum_{j=1}^d \frac{\partial f}{\partial x_j} dx_j.$$ 

So $df$ is a continuous 1-form on $U$. Similarly if $\omega = \sum_S f_S dx_S$ is a $C^1$ $j$-form we define

$$d\omega := \sum_S df_S \wedge dx_S = \sum_S \sum_i \frac{\partial f_S}{\partial x_i} dx_i \wedge dx_S.$$ 

So $d\omega$ is a continuous $j + 1$-form on $U$.

Example 18. Let $\omega = \sum_j f_j dx_j$ be a 1-form. Then

$$d\omega = \sum_{i<j} \left( \frac{\partial f_i}{\partial x_i} - \frac{\partial f_j}{\partial x_j} \right) dx_i \wedge dx_j.$$ 

Example 19. Let $\gamma : [0,1] \to U$ be a 1-surface in $U$, that is a curve in $U$. Suppose $f \in C^1(U)$ then

$$\int_\gamma df = \int_0^1 \sum_{i=1}^k \frac{\partial f}{\partial x_i} dx_i = \int_0^1 \sum_{i=1}^k \frac{\partial f}{\partial x_i} (\gamma(t)) \gamma'_i(t) dt = f(\gamma(1)) - f(\gamma(0)).$$

Lemma 20. Let $\omega$ be a $C^2$ $j$-form on an open set $U \subset \mathbb{R}^d$. Then $d(d\omega) = 0$.

**Proof.** Let $\omega = \sum_S f_S dx_S$. So

$$d(d\omega) = \sum_S \sum_i \sum_j \frac{\partial f}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_S$$

$$= \sum_S \sum_i \sum_{i<j} \left( \frac{\partial f}{\partial x_i x_j} - \frac{\partial f}{\partial x_j x_i} \right) dx_i \wedge dx_j \wedge dx_S = 0.$$ 

Lemma 21. Let $\omega$ be a $C^1$ $j$-form and let $\nu$ be a $C^1$ $k$-form, both defined on an open set $U \subset \mathbb{R}^d$. Then

$$d(\omega \wedge \nu) = (d\omega) \wedge \nu + (-1)^j \omega \wedge (d\nu).$$ 

Exercise 22. Prove Lemma 21. (Hint: it is essentially the product rule.)

Note that the notation $dx_j$ for the elementary one forms is consistent with the exterior derivative: $dx_j$ is indeed the exterior derivative of the function $x \mapsto x_j$. 

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Theorem 23. Let $\omega$ be a $C^1$ $j$-form on an open set $U \subset \mathbb{R}^d$ and let $\Phi : V \to U$ be a parameterized $k$-surface in $U$. Then

$$(d\omega)_\Phi = d(\omega_\Phi).$$

Proof. First consider a zero form, namely a function $f \in C^1(U)$. Then $f_\Phi = f \circ \Phi$ and we see that

$$df_\Phi(y)[v] = df(\Phi(y))[d\Phi(y) \cdot v] = (df)_\Phi(y)[v]$$

by the chain rule. That is,

$$d(f_\Phi) = (df)_\Phi \quad (3)$$

Specializing to a coordinate function $x_j$ we see that

$$(dx_j)_\Phi = d\Phi_j,$$

where $\Phi_j$ denotes the $j$-th coordinate function of $\Phi$. Since $[\omega \wedge \nu]_\Phi = \omega_\Phi \wedge \nu_\Phi$ (see remark 11) we see that

$$(dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_j})_\Phi = d\Phi_{\alpha_1} \wedge \cdots \wedge d\Phi_{\alpha_j}$$

for any elementary $j$-form $dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_j}$. In particular, it follows from the product rule (Lemma 21) that

$$d(dx_S)_\Phi = 0 \quad (4)$$

for any elementary $j$-form $dx_S$.

Putting equations (3) and (4) together we find for a general $j$-form $\omega = \sum_S f_S dx_S$ that

$$(d\omega)_\Phi = \left(\sum_S df_S \wedge dx_S\right)_\Phi = \sum_S (df_S)_\Phi \wedge (dx_S)_\Phi = \sum_S d(f_S \circ \Phi) \wedge (dx_S)_\Phi = d \sum_S (f_S \circ \Phi)(dx_S)_\Phi = d(\omega_\Phi).$$

$\Box$