Integer Division/ Modulo Arithmetic

1. We can add two numbers in base 2 by using the following SUM and CARRY tables. The tables are formed based on mod 2 function; SUM bits of the table correspond to mod 2 of the sum of the two bits being added and the carry bits are the quotients.

<table>
<thead>
<tr>
<th>SUM</th>
<th>CARRY</th>
</tr>
</thead>
<tbody>
<tr>
<td>+ 2</td>
<td>0 1</td>
</tr>
<tr>
<td>----</td>
<td>-------</td>
</tr>
<tr>
<td>0</td>
<td>0 1</td>
</tr>
<tr>
<td>1</td>
<td>1 0</td>
</tr>
</tbody>
</table>

(a) (5 points) Give the SUM and CARRY tables for base 5 addition.

<table>
<thead>
<tr>
<th>SUM</th>
<th>CARRY</th>
</tr>
</thead>
<tbody>
<tr>
<td>+ 5</td>
<td>0 1 2 3 4</td>
</tr>
<tr>
<td>----</td>
<td>----------</td>
</tr>
<tr>
<td>0</td>
<td>0 1 2 3 4</td>
</tr>
<tr>
<td>1</td>
<td>1 2 3 4 0</td>
</tr>
<tr>
<td>2</td>
<td>2 3 4 0 1</td>
</tr>
<tr>
<td>3</td>
<td>3 4 0 1 2</td>
</tr>
<tr>
<td>4</td>
<td>4 0 1 2 3</td>
</tr>
</tbody>
</table>

(b) (5 points) Add the following in base 5:

\[ \begin{array}{c}
\text{231}_5 \\
+ \text{224}_5 \\
\hline
\text{1010}_5 \\
\end{array} \]

2. (15 points) Perform the indicated conversions:

(a) Convert \((210212212)_3\) to base 9: \((23785)_9\)

(b) Convert \((42)_5\) to base 3: \((211)_3\)

(c) Convert \((2.442)_{16}\) to octal:

\(0010\ 1010\ 0100\ 0010\)

\((0)(010)\ \text{(101)}(0\ 01)(00\ 0)(010):\ 025102\)
Matrices

3. (5 points) Compute the boolean product of the following two zero-one matrices:

\[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
1 & 1 \\
0 & 1 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 \\
1 & 1 \\
\end{bmatrix}
\]

Sequences

4. (6 points) What is the nth terms of the following sequence.

5, 10, 20, 40, 80, 160, ..., . . .

5. (2^n) n ≥ 0

5. In showing that the set of all real numbers is **uncountable**, we show that the subset of all real numbers that fall between 0 and 1 is also **uncountable**.

We show this by contradiction assuming that the subset is **countable**. Under this assumption, the real numbers between 0 and 1 can be listed in some order, say \(r_1, r_2, r_3, \ldots\). Let the decimal representation of these numbers be

\[
r_1 = 0.d_{11}d_{12}d_{13}d_{14}... \\
r_2 = 0.d_{21}d_{22}d_{23}d_{24}... \\
r_3 = 0.d_{31}d_{32}d_{33}d_{34}... \\
\ldots
\]

where \(d_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\)

We contradict by defining a real number \(r\) that is between 0 and 1 but not equal to any one of \(r_1, r_2, r_3, \ldots\)

We define

\[
r = 0.d_1d_2d_3d_4... \\
\]

such that

\[
d_i = 4 \text{ if } d_{ii} \neq 4 \\
d_i = 5 \text{ if } d_{ii} = 4
\]

(a) (5 points) What will be the value of \(r\) if

\[
r_1 = 3462... \\
r_2 = 1455... \\
r_3 = 3347... \\
r_4 = 2348... \\
\ldots
\]

\[
r = .4554
\]

(b) (5 points) Indicate why \(r\) is not equal to any of \(r_1, r_2, r_3, \ldots\)

\[d_i \neq d_{ii} \forall i \geq 1\]
(c) (5 points) What assumption guarantees that the proof has considered all the real numbers
between 0 and 1 (i.e., \( r_1, r_2, r_3, \ldots \) covers all reals between 0 and 1)
Assumption that it is a sequence (i.e., all numbers can be presented sequentially in some order)

**Induction**

6. Let \( p(n) \) be a propositional function as follows:

\[ p(n): 1+2+3+4+ \ldots +n = n(n+1)/2 \]

(a) What are

i. (2 points) \( p(2): 1+2=2(2+1)/2 \)

ii. (2 points) \( p(n+1): 1+2+3+4+ \ldots +n+n+1=(n+1)(n+1+1)/2 \)

(b) (2 points) show \( p(2) \)

\[ p(2): 1+2=2(2+1)/2 \]

\[ \iff \ 3=3 \]

\[ \iff \ \text{true} \]

(c) (10 points) Prove \( p(n) \ \forall n \) by mathematical induction.

**Basis Step:**

\[ p(1): 1=1(1+1)/2 \]

\[ 1=1(2)/2 \]

\[ \iff \ 1=1 \]

\[ \iff \ \text{true} \]

**Induction step:**

\[ p(k) \rightarrow p(k+1) \ \forall k > 1 \]

\[ p(k+1): 1+2+3+4+ \ldots +k+(k+1)=(k+1)(k+1+1)/2 \]

By the hypothesis:

\[ k(k+1)/2+(k+1)=(k+1)(k+1+1)/2 \]

By simplifying both sides of the above proposition

\[ (k+1)(k+2)/2=(k+1)(k+2)/2 \]

\[ \iff \ \text{true} \]

7. (8 points) A binary tree is a tree in which no node (vertex) can have more than two children nodes. A full node in a binary tree is a node with two children. Prove by mathematical induction that the number of full nodes plus one is equal to the number of leaves in a non-empty binary tree.

**Theorem:** \( T(N): \) If there are \( N \) full nodes in a non-empty binary tree then there are \( N+1 \) leaves.

**Basis Step:** \( T(0): \) If there are 0 full node in a non-empty binary tree then there is only one leave. This is true because it has only one branch in the tree due to 0 full node.

**Inductive Step:** Show \( T(k) \rightarrow T(k+1) \ \forall k \geq 0 \)
\( T(K + 1) \): If there are \( k+1 \) full nodes in a non-empty binary tree then there are \( k+2 \) leaves.

Pick a leaf node and keep removing it’s parent recursively (i.e., remove its parent and then parent’s parent and so on) until a full node is reached. That is, you are traversing from a leaf along the path towards the root, while removing the nodes along the path before a full node is reached. This full node becomes a non-full node because one of it’s child node is removed. At this point the tree will have one less leaf and one less full node.

Therefore, the tree has \( k \) full nodes after the nodes are removed. By the hypothesis there are \( k+1 \) leaves. Add all the nodes that were removed back into the tree the same way to create the original tree. We are adding one full node and one leaf node. Therefore, we have \( k+1 \) full nodes with \( k+2 \) leaves.

### Recursion

8. Tower of Hanoi Problem is defined as follows:

It consists of a set of disks and three poles. Initially, all disks are on pole one and the task is to move all disks from pole one to pole two, one at a time, using a third pole as a spare. Disks are of different sizes and the rule is that only a smaller disk can be placed on top of a bigger disk. Function, Solve Towers (ST), to solve the problem is recursively defined as follows:

\[
\text{ST(Count, Source, Destination, Spare)} \\
\text{if(Count is 1)} \\
\text{**** Move a disk directly from source to Destination)} \\
\text{Else} \\
\{\text{ST(Count-1, Source, Spare, Destination)} \\
\text{ST(1, Source, Destination, Spare)} \\
\text{ST(Count-1, Spare, Destination, Source)}\}
\]

Consider the function call \( \text{ST}(3,A,B,C) \), i.e., there are 3 disks on pole \( A \) and move them to pole \( B \) using pole \( C \) as the spare.

(a) (10 points) Give the first 6 recursive calls resulting from the initial call \( \text{ST}(3,A,B,C) \)
(b) (5 points) Give the actual moves (resulting from step marked *** above) of the disks from the above recursive calls.
Move top disk from pole A to pole B
Move top disk from pole A to pole C
Move top disk from pole B to pole C

Relation

9. Following directed graph (digraph) represents a relation R defined over the set $S = \{A, B, C, D\}$.

(a) (5 points) Write the relation R as a set of elements of $S \times S$.
\{(A,A), (A,B), (C,B), (B,B), (C,C), (C,D), (D,D)\}

(b) (5 points) Circle the appropriate properties below for the relation.
- reflexive(X) irreflexive symmetric antisymmetric(X) asymmetric transitive(X)