### Asymptotic Notation Chart:

<table>
<thead>
<tr>
<th>if</th>
<th>the growth-rate is</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(n) = \Theta(g(n))$</td>
<td>$f(n) = g(n)$</td>
</tr>
<tr>
<td>$f(n) = O(g(n))$</td>
<td>$f(n) \leq g(n)$</td>
</tr>
<tr>
<td>$f(n) = \Omega(g(n))$</td>
<td>$f(n) \geq g(n)$</td>
</tr>
<tr>
<td>$f(n) = o(g(n))$</td>
<td>$f(n) &lt; g(n)$</td>
</tr>
<tr>
<td>$f(n) = \omega(g(n))$</td>
<td>$f(n) &gt; g(n)$</td>
</tr>
</tbody>
</table>

$f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n))$
Prove that: \( f(n) = \Theta(g(n)) \)

Where: \( f(n) = 100n^2 + 10n \)
Where: \( g(n) = n^2 + n \)

In order to prove

\[ f(n) = \Theta(g(n)) \]

We need to prove BOTH

\[ f(n) = O(g(n)) \]
\[ f(n) \leq \Omega(g(n)) \]

\[ f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n)) \]
We will prove this first:

\[ f(n) = O(g(n)) \]

Let \( C = 200 \) and \( N_0 = 1 \)

Then \( 100n^2 + 10n \leq C(n^2 + n) \)

\[ = 200n^2 + 200n \quad \text{For } N \geq N_0 \rightarrow N \geq 1 \]

When proving: \( f(n) = O(g(n)) \)

Choose a value for \( C > 1 \)

\[ f(n) = 100n^2 + 10n \]
\[ g(n) = n^2 + n \]
Now we prove:

\[ f(n) = f(n) = \Omega(g(n)) \]

Let \( C = 0.5 \) and \( N_0 = 1 \)

Then \( 100n^2 + 10n \geq C(n^2 + n) \)

\[ = 0.5n^2 + 0.5n \quad \text{For } N \geq N_0 \rightarrow N \geq 1 \]

When proving: \( f(n) = \Omega(g(n)) \)

Choose a value for \( C < 1 \)

Using our two proofs:

\[ f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n)) \]

\[ f(n) = 100n^2 + 10n \]

\[ g(n) = n^2 + n \]
Insertion Sort:

1. What are the correct intermediate steps of the following data set when it is being sorted with the Insertion sort? 15,20,10,18
   A. 15,20,10,18 -- 10,15,20,18 -- 10,15,18,20 -- 10,15,18,20
   B. 15,18,10,20 -- 10,18,15,20 -- 10,15,18,20 -- 10,15,18,20
   C. 15,10,20,18 -- 15,10,18,20 -- 10,15,18,20
   D. 10,20,15,18 -- 10,15,20,18 -- 10,15,18,20

Step 0 (initially): 15,20,10,18

Step 1: 15,20,10,18   Final Data Set: 10,15,18,20

Step 2: 15,20,10,18

Step 3: 10,15,20,18

Step 4: 10,15,18,20

Another way to think about it:

DataSet: [15,20,10,18]
Sorted Array: []
DataSet: [20,10,18]
Sorted Array: [15]
DataSet: [10,18]
Sorted Array: [15,20]
DataSet: [18]
Sorted Array: [10,15,20]
DataSet: []
Sorted Array: [10,15,18,20]

Answer: A
One more on Insertion Sort:

2. Consider the following lists of partially sorted numbers. The double bars represent the sort marker. How many comparisons and swaps are needed to sort the next number.

\[1 \ 3 \ 4 \ 8 \ 9 \ || \ 5 \ 2\]

A. 2 comparisons, 3 swaps
B. 3 comparisons, 2 swaps
C. 4 comparisons, 3 swaps
D. 3 comparisons, 4 swaps

Answer: B
Q-17: Given the following list of numbers: [21, 1, 26, 45, 29, 28, 2, 9, 16, 49, 39, 27, 43, 34, 46, 40] which answer illustrates the list to be sorted after 3 recursive calls to mergesort?

(A) [16, 49, 39, 27, 43, 34, 46, 40]
(B) [21, 1]
(C) [21, 1, 26, 45]
(D) [21]

First Call: [21, 1, 26, 45, 29, 28, 2, 9] [16, 49, 39, 27, 43, 34, 46, 40]
Second Call: [21, 1, 26, 45] [29, 28, 2, 9] [16, 49, 39, 27] [43, 34, 46, 40]
Third Call: [21, 1] [26, 45] [29, 28] [2, 9] [16, 49] [39, 27] [43, 34] [46, 40]

Answer: B
Master Theorem:

The Master Theorem applies to recurrences of the following form:

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is an asymptotically positive function.

There are 3 cases:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.

2. If $f(n) = \Theta(n^{\log_b a \log^k n})$ with $k \geq 0$, then $T(n) = \Theta(n^{\log_b a \log^{k+1} n})$.

3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ with $\epsilon > 0$, and $f(n)$ satisfies the regularity condition, then $T(n) = \Theta(f(n))$.

   Regularity condition: $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large $n$. 
Master Theorem:

1. If $f(n) = O(n^{\log_b a-\epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a \log^k n})$ with $k \geq 0$, then $T(n) = \Theta(n^{\log_b a \log^{k+1} n})$.
3. If $f(n) = \Omega(n^{\log_b a+\epsilon})$ with $\epsilon > 0$, and $f(n)$ satisfies the regularity condition, then $T(n) = \Theta(f(n))$.

Regularity condition: $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large $n$.

Which Case?

\[
T(n) = aT(n/b) + f(n)
\]

- $T(n) = 3T(n/3) + n/2 \implies T(n) = \Theta(n \log n)$ (Case 2)
- $T(n) = 6T(n/3) + n^2 \log n \implies T(n) = \Theta(n^2 \log n)$ (Case 3)
- $T(n) = 4T(n/2) + n/ \log n \implies T(n) = \Theta(n^2)$ (Case 1)
Master Theorem:  

\[ T(n) = aT(n/b) + f(n) \]

\[ T(n) = 3T(n/3) + n/2 \implies T(n) = \Theta(n \log n) \text{ (Case 2)} \]

What we Know:

- \( f(n) = n/2 \)
- \( a = 3 \)
- \( b = 3 \)

Is it case 1?

1. If \( f(n) = O(n^{\log_b a - \epsilon}) \) for some constant \( \epsilon > 0 \), then \( T(n) = \Theta(n^{\log_b a}) \).

\[ \frac{n}{2} = O(n^{\log_3 (3 - 1)})? \]

Let \( \epsilon = 1 \)

\[ \frac{n}{2} = O(n^{\log_3 (3 - 1)})? \quad \text{No, not case 1} \]

Is it case 2?

\[ \frac{n}{2} = \Theta(n^{\log_b (a) \log(n)^k})? \quad k \geq 0 \]

\[ \frac{n}{2} = \Theta(n^{\log_3 (3) \log(n)^k})? \quad \log_3 (3) = 1 \]

\[ \frac{n}{2} = \Theta(n^{1 \log(n)^0})? \quad k = 0 \]

\[ \frac{n}{2} = \Theta(n)? \quad \text{Yes, case 2} \]

Therefore:

\[ T(n) = \Theta(n \log n) \]

2. If \( f(n) = \Theta(n^{\log_b a \log^k n}) \) with \(^1\ k \geq 0\), then \( T(n) = \Theta(n^{\log_b a \log^{k+1} n}) \).
Master Theorem:

\[ T(n) = aT(n/b) + f(n) \]

\[ T(n) = 6T(n/3) + n^2 \log n \implies T(n) = \Theta(n^2 \log n) \quad \text{(Case 3)} \]

What we Know:

- \( f(n) = (n^2)\log(n) \)
- \( a = 6 \)
- \( b = 3 \)

Is it case 1?  
1. If \( f(n) = O(n^{\log_b a - \epsilon}) \) for some constant \( \epsilon > 0 \), then \( T(n) = \Theta(n^{\log_b a}) \).

- \((n^2)\log(n) = O(n^{(\log_b 6 - \epsilon)})? \) Let \( \epsilon = 3 \)
  - No, not case 1

Is it case 2?  
2. If \( f(n) = \Theta(n^{\log_b a}) \log(n)^k \log(n)^0 \).

- \((n^2)\log(n) = \Theta(n^{\log_3 6})\log(n)^k? \) \( k \geq 0 \)
  - No, not case 2

Is it case 3?  
3. If \( f(n) = \Omega(n^{\log_b a + \epsilon}) \) with \( \epsilon > 0 \), and \( f(n) \) satisfies the regularity condition, then \( T(n) = \Theta(f(n)) \).

- Regularity condition: \( 6f(n/3) \leq cf(n) \) \( c = 0.5 \) \( n = 3 \)
  - \( 6\log(1) \leq (0.5)(9)\log(3)? \) Yes

3. If \( f(n) = \Omega(n^{\log_b a + \epsilon}) \) with \( \epsilon > 0 \), and \( f(n) \) satisfies the regularity condition, then \( T(n) = \Theta(f(n)) \).

  Regularity condition: \( af(n/b) \leq cf(n) \) for some constant \( c < 1 \) and all sufficiently large \( n \).
Master Theorem:

\[ T(n) = aT\left(\frac{n}{b}\right) + f(n) \]

\[ T(n) = 6T\left(\frac{n}{3}\right) + n^2 \log n \implies T(n) = \Theta(n^2 \log n) \] (Case 3)

What we Know:

- \( a = 6 \)
- \( b = 3 \)
- \( f(n) = (n^2)\log(n) \)

Is it case 3? Regularity condition: \( 6f(n/3) \leq cf(n) \)

\[ c = 0.5 \quad n = 3 \]

\[ 6\log(1) \leq (0.5)(9)\log(3)? \quad \text{Yes} \]

Let \( \epsilon = 3 \)

\[ (n^2)\log(n) = \Omega(n^{\log_3(6 - 3)})? \]

\[ (n^2)\log(n) = \Omega(n^{\log_3(6 - 3)})? \quad \log_3(3) = 1 \]

\[ (n^2)\log(n) = \Omega(n)? \quad \text{Yes, case 3} \]

Therefore: \( T(n) = \Theta(f(n)) \implies T(n) = \Theta((n^2)\log(n)) \)

3. If \( f(n) = \Omega(n^{\log_b a + \epsilon}) \) with \( \epsilon > 0 \), and \( f(n) \) satisfies the regularity condition, then \( T(n) = \Theta(f(n)) \).

Regularity condition: \( af(n/b) \leq cf(n) \) for some constant \( c < 1 \) and all sufficiently large \( n \).
Master Theorem: 

\[ T(n) = aT(n/b) + f(n) \]

\[ T(n) = 4T(n/2) + n/\log n \implies T(n) = \Theta(n^2) \text{ (Case 1)} \]

What we Know: 

- \( f(n) = n/\log(n) \)
- \( a = 4 \)
- \( b = 2 \)

Is it case 1?

- \( n/\log(n) = O(n^{\log_b(a-\epsilon)})? \)
- Let \( \epsilon = 2 \)
- \( n/\log(n) = O(n^{\log_2(4-2)})? \)
- \( n/\log(n) = O(n)? \)
- Yes, case 1

Therefore:

\[ T(n) = \Theta(n^{\log_b(a)}) \]
\[ T(n) = \Theta(n^{\log_2(4)}) \]
\[ T(n) = \Theta(n^2) \]

1. If \( f(n) = O(n^{\log_b a - \epsilon}) \) for some constant \( \epsilon > 0 \), then \( T(n) = \Theta(n^{\log_b a}) \).
Heap:

Build the min heap resulting from inserting 11, 9, 12, 14, 3, 15, 7, 8, 1