Outline

1. Greedy Algorithms
2. Elements of Greedy Algorithms
3. Greedy Choice Property for Kruskal’s Algorithm
4. 0/1 Knapsack Problem
5. Activity Selection Problem
6. Scheduling All Intervals
Greedy algorithms is another useful way for solving optimization problems.

Optimization Problems

- For the given input, we are seeking solutions that must satisfy certain conditions.
- These solutions are called feasible solutions. (In general, there are many feasible solutions.)
- We have an optimization measure defined for each feasible solution.
- We are looking for a feasible solution that optimizes (either maximum or minimum) the optimization measure.
Matrix Chain Product Problem

- A feasible solution is any valid parenthesization of an $n$-term chain.
- The optimization measure is the total number of scalar multiplications for the parenthesization.
- Goal: Minimize the total number of scalar multiplications.
Examples

Matrix Chain Product Problem
- A feasible solution is any valid parenthesization of an $n$-term chain.
- The optimization measure is the total number of scalar multiplications for the parenthesization.
- Goal: Minimize the total number of scalar multiplications.

0/1 Knapsack Problem
- A feasible solution is any subset of items whose total weight is at most the knapsack capacity $K$.
- The optimization measure is the total item profit of the subset.
- Goal: Maximize the total profit.
Greedy Algorithms

General Description

- Given an optimization problem $P$, we seek an optimal solution.
- The solution is obtained by a sequence of steps.
- In each step, we select an “item” to be included into the solution.
- At each step, the decision is made based on the selections we have already made so far, that looks the best choice for achieving the optimization goal.
- Once a selection is made, it cannot be undone: The selected item cannot be removed from the solution.
Minimum Spanning Tree (MST) Problem

This is a classical graph problem. We will study graph algorithms in detail later. Here we use MST as an example of Greedy Algorithms.

**Definition**

A **tree** is a connected graph with no cycles.

**Definition**

Let \( G = (V, E) \) be a graph. A **spanning tree** of \( G \) is a subgraph of \( G \) that contains all vertices of \( G \) and is a tree.

**Minimum Spanning Tree (MST) Problem**

Input: An connected undirected graph \( G = (V, E) \). Each edge \( e \in E \) has a weight \( w(e) \geq 0 \).

Find: a spanning tree \( T \) of \( G \) such that \( w(T) = \sum_{e \in T} w(e) \) is minimum.
Kruskal’s Algorithm

1: Sort the edges by non-decreasing weight. Let $e_1, e_2, \ldots, e_m$ be the sorted edge list
2: $T \leftarrow \emptyset$
3: for $i = 1$ to $m$ do
4:  if $T \cup \{e_i\}$ does not contain a cycle then
5:     $T \leftarrow T \cup \{e_i\}$
6:  else
7:     do nothing
8:  end if
9: end for
10: output $T$
Kruskal’s Algorithm

- The algorithm goes through a sequence of steps.
- At each step, we consider the edge $e_i$, and decide whether add $e_i$ into $T$.
- Since we are building a spanning tree $T$, $T$ can not contain any cycle. So if adding $e_i$ into $T$ introduces a cycle in $T$, we do not add it into $T$.
- Otherwise, we add $e_i$ into $T$. We are processing the edges in the order of increasing edge weight. So when $e_i$ is added into $T$, it looks the best to achieve the goal (minimum total weight).
- Once $e_i$ is added, it is never removed and is included into the final tree $T$.
- This is a perfect example of greedy algorithms.
The number near an edge is its weight. The blue edges are in the MST constructed by Kruskal’s algorithm.

The blue numbers in () indicate the order in which the edges are added into MST.
Kruskal’s Algorithm

- For a given graph $G = (V, E)$, its MST is not unique. However, the weight of any two MSTs of $G$ must be the same.
- In Kruskal’s algorithm, two edges $e_i$ and $e_{i+1}$ may have the same weight. If we process $e_{i+1}$ before $e_i$, we may get a different MST.
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- Runtime of Kruskal’s algorithm:
  - Sorting of edge list takes $\Theta(m \log m)$ time.
  - Then we process the edges one by one. So the loop iterates $m$ time.
  - When processing an edge $e_i$, we check if $T \cup \{e_i\}$ contains a cycle or not. If not, add $e_i$ into $T$. If yes, do nothing.
  - By using disjoint-set data structure, the processing of an edge $e_i$ can be done in $O(\log n)$ time on average.
  - So the loop takes $O(m \log n)$ time.
  - Since $G$ is connected, $m \geq n$. The total runtime is $\Theta(m \log m + m \log n) = \Theta(m \log m)$. 
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2. Elements of Greedy Algorithms

3. Greedy Choice Property for Kruskal’s Algorithm

4. 0/1 Knapsack Problem

5. Activity Selection Problem

6. Scheduling All Intervals
Elements of Greedy Algorithms

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- In this case, we are lucky: our intuition is correct.
- But in other cases, the strategies that seem equally obvious may lead to wrong solutions.
- In general, the correctness of a greedy algorithm requires proof.
Correctness Proof of Algorithms

- An algorithm $A$ is **correct**, if it works on **all** inputs.
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- If $A$ works on some inputs, but not on some other inputs, then $A$ is incorrect.
- To show $A$ is correct, you must argue that for all inputs, $A$ produces intended solution.

Strictly speaking, all algorithms need correctness proof. For DaC, it's often so straightforward that the correctness proof is unnecessary/omitted. (Example: MergeSort) For dynamic programming algorithms, the correctness proof is less obvious than the DaC algorithms. But in most time, it is quite easy to convince people (i.e. informal proof) the algorithm is correct. For greedy algorithms, the correctness proof can be very tricky.
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- An algorithm $A$ is **correct**, if it works on all inputs.

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- Strictly speaking, all algorithms need correctness proof.
- For DaC, it’s often so straightforward that the correctness proof is unnecessary/omitted. (Example: MergeSort)
- For dynamic programming algorithms, the correctness proof is less obvious than the DaC algorithms. But in most time, it is quite easy to convince people (i.e. informal proof) the algorithm is correct.
- For greedy algorithms, the correctness proof can be very **tricky**.
For a greedy strategy to work, it must have the following two properties.

Optimal Substructure Property
An optimal solution of the problem contains within it the optimal solutions of subproblems.

Greedy Choice Property
A global optimal solution can be obtained by making a locally optimal choice that seems the best toward the optimization goal when the choice is made. (Namely: The choice is made based on the choices we have already made, not based on the future choices we might make.)

This property is harder to describe exactly. The best way to understand it is by examples.
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Example

Optimal Substructure Property for MST

- Let $G = (V, E)$ be a connected graph with edge weight.
- Let $e_1 = (x, y)$ be the edge with the smallest weight. (Namely, $e_1$ is the first edge chosen by Kruskal’s algorithm.)
- Let $G' = (V', E')$ be the graph obtained from $G$ by merging $x$ and $y$:
  - $x$ and $y$ becomes a single new vertex $z$ in $G'$.
  - Namely $V' = V - \{x, y\} \cup \{z\}$
  - $e_1$ is deleted from $G$.
  - Any edge $e_i$ in $G$ that was incident to $x$ or $y$ now is incident to $z$.
  - The edge weights remain unchanged.
Optimal Substructure Property for MST

If $T$ is a MST of $G$ containing $e_1$, then $T' = T - \{e_1\}$ is a MST of $G'$.
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Let $e_1, e_2, \ldots, e_m$ be the edge list in the order of increasing weight. So $e_1$ is the first edge chosen by Kruskal’s algorithm.

Let $T_{opt}$ be an MST of $G$. By definition, the total weight of $T_{opt}$ is the minimum.

We want to show $T_{opt}$ contains $e_1$.

But this is not always possible. Recall that the MST of $G$ is not unique.

So we will do this: Starting from $T_{opt}$, we change $T_{opt}$, without increasing the weight in the process, to another MST $T'$ that contains $e_1$.

If $T_{opt}$ contains $e_1$, then we are done (lucky!)
Greedy Choice Property for Kruskal’s Algorithm

- Suppose $T_{opt}$ does not contain $e_1$.
- Consider the graph $H = T_{opt} \cup \{e_1\}$.
- $H$ contains a cycle $C$. Let $e_i \neq e_1$ be another edge on $C$.
- Let $T' = T_{opt} - \{e_i\} \cup \{e_1\}$.
- Then $T'$ is a spanning tree of $G$.
- Since $e_1$ is the edge with the smallest weight, $w(e_1) \leq w(e_i)$.
- Hence $w(T') = w(T_{opt}) - w(e_i) + w(e_1) \leq w(T_{opt})$.
- But $T_{opt}$ is a MST!
- So we must have $w(e_i) = w(e_1)$ and $w(T_{opt}) = w(T')$. In other words, both $T_{opt}$ and $T'$ are MSTs of $G$.
- This is what we want to show: There is an MST that contains $e_1$. So when Kruskal’s algorithm includes $e_1$ into $T$, we are not making a mistake.
Greedy Choice Property for Kruskal’s Algorithm
The proof is by induction.
Correctness Proof of Kruskal’s Algorithm

- The proof is by induction.
- Kruskal’s algorithm selects the lightest edge \( e_1 = (x, y) \).
Correctness Proof of Kruskal’s Algorithm

- The proof is by induction.
- Kruskal’s algorithm selects the lightest edge $e_1 = (x, y)$.
- By Greedy Choice Property, there exists an optimal MST of $G$ that contains $e_1$. 
Correctness Proof of Kruskal’s Algorithm

The proof is by induction.

Kruskal’s algorithm selects the lightest edge $e_1 = (x, y)$.

By Greedy Choice Property, there exists an optimal MST of $G$ that contains $e_1$.

By induction hypothesis, Kruskal’s algorithm construct a MST $T'$ in the graph $G' = ((V - \{x, y\} \cup \{z\}), E')$ which is obtained from $G$ by merging the two end vertices $x, y$ of $e_1$. 
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By the **Optimal Substructure Property of MST**, $T = T' \cup \{e_1\}$ is a MST of $G$. 

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Correctness Proof of Kruskal’s Algorithm

- The proof is by induction.
- Kruskal’s algorithm selects the lightest edge $e_1 = (x, y)$.
- By *Greedy Choice Property*, there exists an optimal MST of $G$ that contains $e_1$.
- By induction hypothesis, Kruskal’s algorithm construct a MST $T'$ in the graph $G' = ((V - \{x, y\} \cup \{z\}), E')$ which is obtained from $G$ by merging the two end vertices $x, y$ of $e_1$.
- By the *Optimal Substructure Property of MST*, $T = T' \cup \{e_1\}$ is a MST of $G$.
- This $T$ is the tree constructed by Kruskal’s algorithm. Hence, Kruskal’s algorithm indeed returns a MST.
We mentioned that some seemingly intuitive greedy strategies do not really work. Here is an example.

Input: $n$ item $i$ ($1 \leq i \leq n$). Each item $i$ has an integer weight $w[i] \geq 0$ and a profit $p[i] \geq 0$.
A knapsack with an integer capacity $K$.
Find: A subset of items so that the total weight of the selected items is at most $K$, and the total profit is maximized.

There are several greedy strategies that seem reasonable. But none of them works.
Greedy Strategy 1

Since the goal is to maximize the profit without exceeding the capacity, we fill the items in the order of increasing weights. Namely:

- Sort the items by increasing item weight: \( w[1] \leq w[2] \leq \cdots \).
- Fill the knapsack in the order \( \text{item}_1, \text{item}_2, \ldots \) until no more items can be put into the knapsack without exceeding the capacity.
0/1 Knapsack Problem

Greedy Strategy 1
Since the goal is to maximize the profit without exceeding the capacity, we fill the items in the order of increasing weights. Namely:

- Sort the items by increasing item weight: \( w[1] \leq w[2] \leq \cdots \).
- Fill the knapsack in the order item\(_1\), item\(_2\), ... until no more items can be put into the knapsack without exceeding the capacity.

Counter Example:
- This strategy puts item\(_1\) into the knapsack with total profit 2.
- The optimal solution: put item\(_2\) into the knapsack with total profit 3.
For this greedy strategy, we can still show the **Optimal Substructure Property** holds:

- if $S$ is an optimal solution, that contains the item 1, for the original input,
For this greedy strategy, we can still show the **Optimal Substructure Property** holds:

- if $S$ is an optimal solution, that contains the item $1$, for the original input,
- then $S - \{\text{item}_1\}$ is an optimal solution for the input consisting of $\text{item}_2$, $\text{item}_3$, $\cdots$, $\text{item}_n$ and the knapsack with capacity $K - w[1]$. 
For this greedy strategy, we can still show the **Optimal Substructure Property** holds:

- if \( S \) is an optimal solution, that contains the item\(_1\), for the original input,
- then \( S - \{\text{item}_1\} \) is an optimal solution for the input consisting of \( \text{item}_2, \text{item}_3, \cdots, \text{item}_n \) and the knapsack with capacity \( K - w[1] \).

However, we cannot prove the **Greedy Choice Property**: We are not able to show there is an optimal solution that contains the item\(_1\) (the lightest item).
For this greedy strategy, we can still show the **Optimal Substructure Property** holds:

- if $S$ is an optimal solution, that contains the item $1$, for the original input,
- then $S \setminus \{\text{item}_1\}$ is an optimal solution for the input consisting of item $2$, item $3$, \ldots, item $n$ and the knapsack with capacity $K - w[1]$.

However, we cannot prove the **Greedy Choice Property**: We are not able to show there is an optimal solution that contains the item $1$ (the lightest item).

Without this property, there is no guarantee this strategy would work. (As the counter example has shown, it doesn’t work.)
Greedy Strategy 2

Since the goal is to maximize the profit without exceeding the capacity, we fill the items in the order of decreasing profits. Namely:

- Sort the items by decreasing item profit: \( p[1] \geq p[2] \geq \cdots \).
- Fill the knapsack in the order item \( 1 \), item \( 2 \), ... until no more items can be put into the knapsack without exceeding the capacity.
Greedy Strategy 2

Since the goal is to maximize the profit without exceeding the capacity, we fill the items in the order of decreasing profits. Namely:

- Sort the items by decreasing item profit: \( p[1] \geq p[2] \geq \cdots \).
- Fill the knapsack in the order item_1, item_2, ... until no more items can be put into the knapsack without exceeding the capacity.

Counter Example:


- This strategy puts item_1 into the knapsack with total profit 3.
- The optimal solution: put item_2 and item_3 into the knapsack with total profit 4.
Greedy Strategy 3

Since the goal is to maximize the profit without exceeding the capacity, we fill the items in the order of decreasing unit profit. Namely:

- Sort the items by decreasing item unit profit: \( \frac{p[1]}{w[1]} \geq \frac{p[2]}{w[2]} \geq \frac{p[3]}{w[1]} \cdots \)

- Fill the knapsack in the order item \( 1 \), item \( 2 \), ... until no more items can be put into the knapsack without exceeding the capacity.
Knapsack Problem

Greedy Strategy 3

Since the goal is to maximize the profit without exceeding the capacity, we fill the items in the order of decreasing unit profit. Namely:

- Sort the items by decreasing item unit profit: \( \frac{p[1]}{w[1]} \geq \frac{p[2]}{w[2]} \geq \frac{p[3]}{w[1]} \cdots \)
- Fill the knapsack in the order item \(1\), item \(2\), ... until no more items can be put into the knapsack without exceeding the capacity.

Counter Example:


- We have: \( \frac{p[1]}{w[1]} = \frac{2}{2} = 1 \geq \frac{p[2]}{w[2]} = \frac{3}{4} \).
- This strategy puts item \(1\) into knapsack with total profit 2.
- The optimal solution: put item \(2\) into knapsack with total profit 3.
Fractional Knapsack Problem

Input: $n$ items ($1 \leq i \leq n$). Each item $i$ has an integer weight $w[i] \geq 0$ and a profit $p[i] \geq 0$.

A knapsack with an integer capacity $K$.

Find: A subset of items to put into the knapsack. **We can select a fraction of an item.** The goal is the same: the total weight of the selected items is at most $K$, and the total profit is maximized.
Fractional Knapsack Problem

Input: $n$ item $i$ ($1 \leq i \leq n$). Each item $i$ has an integer weight $w[i] \geq 0$ and a profit $p[i] \geq 0$.
A knapsack with an integer capacity $K$.
Find: A subset of items to put into the knapsack. We can select a fraction of an item. The goal is the same: the total weight of the selected items is at most $K$, and the total profit is maximized.

Mathematical description of Fractional Knapsack Problem

Input: $2n + 1$ integers $p[1], p[2], \ldots, p[n], \ w[1], w[2], \ldots, w[n], \ K$
Find: a vector $(x_1, x_2, \ldots, x_n)$ such that:

- $0 \leq x_i \leq 1$ for $1 \leq i \leq n$
- $\sum_{i=1}^{n} x_i \cdot w[i] \leq K$
- $\sum_{i=1}^{n} x_i \cdot p[i]$ is maximized.
Although the Fractional Knapsack Problem looks very similar to the 0/1 Knapsack Problem, it is much much easier.

```
1: Sort the items by decreasing unit profit:
2: i = 1 
3: while K > 0 do 
4: if K > w[i] then 
5: x[i] = 1 and K = K − w[i] 
6: else 
7: x[i] = K / w[i] and K = 0 
8: end if 
9: i = i + 1 
10: end while 
```

It can be shown the Greedy Choice Property holds in this case.
Although the **Fractional Knapsack Problem** looks very similar to the **0/1 Knapsack Problem**, it is much much easier.

The **Greedy Strategy** 3 works.
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The Greedy Strategy 3 works.

**Greedy-Fractional-Knapsack**

1: Sort the items by decreasing unit profit: $\frac{p[1]}{w[1]} \geq \frac{p[2]}{w[2]} \geq \frac{p[3]}{w[3]} \cdots$
2: $i = 1$
3: while $K > 0$ do
4: \hspace{1em} if $K > w[i]$ then
5: \hspace{2em} $x_i = 1$ and $K = K - w[i]$
6: \hspace{1em} else
7: \hspace{2em} $x_i = K/w[i]$ and $K = 0$
8: \hspace{1em} end if
9: \hspace{1em} $i = i + 1$
10: end while

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1. Sort the items by decreasing unit profit: \( \frac{p[1]}{w[1]} \geq \frac{p[2]}{w[2]} \geq \frac{p[3]}{w[3]} \cdots \)
2. \( i = 1 \)
3. **while** \( K > 0 \) **do**
4. \( \text{if } K > w[i] \text{ then} \)
5. \( x_i = 1 \text{ and } K = K - w[i] \)
6. \( \text{else} \)
7. \( x_i = K/w[i] \text{ and } K = 0 \)
8. **end if**
9. \( i = i + 1 \)
10. **end while**

It can be shown the Greedy Choice Property holds in this case.
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Activity Selection Problem

A set $S = \{1, 2, \ldots, n\}$ of activities.

Each activity $i$ has a starting time $s_i$ and a finishing time $f_i$ ($s_i \leq f_i$).

Two activities $i$ and $j$ are compatible if the interval $[s_i, f_i)$ and $[s_j, f_j)$ do not overlap.

Goal: Select a subset $A \subseteq S$ of mutually compatible activities so that $|A|$ is maximized.

Application

Consider a single CPU computer. It can run only one job at any time.

Each activity $i$ is a job to be run on the CPU that must start at time $s_i$ and finish at time $f_i$.

How to select a maximum subset $A$ of jobs to run on CPU?
Greedy Algorithm for Activity Selection Problem

Greedy Strategy

At any moment \( t \), select the activity \( i \) with the smallest finish time \( f_i \).

```
Greedy-Activity-Selection
1: Sort the activities by increasing finish time: \( f_1 \leq f_2 \leq \cdots \leq f_n \)
2: \( A = \{1\} \) (\( A \) is the set of activities to be selected.)
3: \( j = 1 \) (\( j \) is the current activity being considered.)
4: for \( i = 2 \) to \( n \) do
5: if \( s_i \geq f_j \) then
6: \( A = A \cup \{i\} \)
7: \( j = i \)
8: end if
9: end for
10: return \( A \)
```
Greedy Algorithm for Activity Selection Problem

Greedy Strategy
At any moment $t$, select the activity $i$ with the smallest finish time $f_i$.

Greedy-Activity-Selection

1: Sort the activities by increasing finish time: $f_1 \leq f_2 \leq \cdots \leq f_n$
2: $A = \{1\}$ ($A$ is the set of activities to be selected.)
3: $j = 1$ ($j$ is the current activity being considered.)
4: for $i = 2$ to $n$ do
5:   if $s_i \geq f_j$ then
6:      $A = A \cup \{i\}$
7:   $j = i$
8:   end if
9: end for
10: return $A$
Dashed lines are not selected
Solid lines are selected activities
After Sorting

Input

After Sorting

Solid lines are selected activities
Dashed lines are not selected

This problem is also called the interval scheduling problem.
Example

[1, 3) is the first interval selected. The dashed intervals [0, 4) and [2, 6) are killed because they are not compatible with [1, 3).
[1, 3) is the first interval selected. The dashed intervals [0, 4) and [2, 6) are killed because they are not compatible with [1, 3).

This problem is also called the interval scheduling problem.
Proof of Correctness

Let $S = \{1, 2, \ldots, n\}$ be the set of activities to be selected. Assume $f_1 \leq f_2 \leq \cdots f_n$. 

**Greedy Choice Property**

The activity 1 is selected by the greedy algorithm. We need to show there is an optimal solution that contains the activity 1.

If the optimal solution $O$ contains 1, we are done.

If not, let $k$ be the first activity in $O$. Let $O' = O - \{k\} \cup \{1\}$.

Since $f_1 \leq f_k$, all activities in $O'$ are still mutually compatible.

Clearly $|O| = |O'|$. So $O'$ is an optimal solution containing 1.
Let $S = \{1, 2, \ldots, n\}$ be the set of activities to be selected. Assume $f_1 \leq f_2 \leq \cdots f_n$.

Let $O$ be an optimal solution. Namely $O$ is a subset of mutually compatible activities and $|O|$ is maximum.
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- Let $X$ be the output from the Greedy algorithm. We always have $1 \in X$. 

Greedy Choice Property

The activity 1 is selected by the greedy algorithm. We need to show there is an optimal solution that contains the activity 1.

If the optimal solution $O$ contains 1, we are done.

If not, let $k$ be the first activity in $O$. Let $O' = O \setminus \{k\} \cup \{1\}$.

Since $f_1 \leq f_k$, all activities in $O'$ are still mutually compatible.

Clearly $|O| = |O'|$. So $O'$ is an optimal solution containing 1.
Proof of Correctness

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By the **Greedy Choice Property**, we may assume the optimal solution $O$ contains the job 1.
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### Optimal Substructure Property

Let $S_1 = \{ i \in S \mid s_i \geq f_1 \}$. ($S_1$ is the set of jobs that are compatible with job 1. Or equivalently, the set of jobs that are not killed by job 1.)
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- Let $O_1 = O - \{1\}$. 

Claim: $O_1$ is an optimal solution of the job set $S_1$.

If this is not true, let $O'_1$ be an optimal solution set of $S_1$. Since $O_1$ is not optimal, we have $|O'_1| > |O_1|$. Let $O''_1 = O'_1 \cup \{1\}$. Then $O''_1$ is a set of mutually compatible jobs in $S$, and $|O''_1| = |O'_1| + 1 > |O_1| + 1 = |O|$. But $O$ is an optimal solution. This is a contradiction. Hence the claim is true.
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Proof of Correctness

Jobs in $S_1$
Proof of Correctness

Since the **Optimal Substructure** and **Greedy Choice** properties are true, we can prove the correctness of the greedy algorithm by induction.

- The greedy algorithm picks the job 1 in its solution.
- By the Greedy Choice property, there is an optimal solution that also contains the job 1. So this selection needs not be reversed.
- The greedy algorithm deletes all jobs that are incompatible with job 1. The remaining jobs is the set $S_1$ in the proof of Optimal Substructure property.
- By induction hypothesis, Greedy algorithm will output an optimal solution $X_1$ for $S_1$.
- By the Optimal Substructure property, $X = X_1 \cup \{1\}$ is an optimal solution of the original job set $S$.
- $X$ is the output from Greedy algorithm. So the algorithm is correct.

**Runtime:** Clearly $O(n \log n)$ (dominated by sorting).
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1. Greedy Algorithms
2. Elements of Greedy Algorithms
3. Greedy Choice Property for Kruskal’s Algorithm
4. 0/1 Knapsack Problem
5. Activity Selection Problem
6. Scheduling All Intervals
Scheduling All Intervals

- Schedule all activities using as few resources as possible.
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**Input:**
- A set \( R = \{I_1, \ldots, I_n\} \) of \( n \) requests/activities.
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- Schedule all activities using as few resources as possible.

**Input:**

- A set $\mathcal{R} = \{I_1, \ldots, I_n\}$ of $n$ requests/activities.
- Each $I_i$ has a start time $s_i$ and finish time $f_i$. (So each $I_i$ is represented by an interval $[s_i, f_i]$).
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**Output:** A partition of $\mathcal{R}$ into as few subsets as possible, so that the intervals in each subset are mutually compatible. (Namely, they do not overlap.)
Scheduling All Intervals

Application

- Each request $I_i$ is a job to be run on a CPU.
Scheduling All Intervals

Application

- Each request $I_i$ is a job to be run on a CPU.
- If two intervals $I_p$ and $I_q$ overlap, they cannot run on the same CPU.
Scheduling All Intervals

Application

- Each request $I_i$ is a job to be run on a CPU.
- If two intervals $I_p$ and $I_q$ overlap, they cannot run on the same CPU.
- How to run all jobs using as few CPUs as possible?
Another way to look at the problem:
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- Color the intervals in $\mathcal{R}$ by different colors.
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- Using as few colors as possible.

This problem is also known as the Interval Graph Coloring Problem.
Scheduling All Intervals

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Graph Coloring

Let $G = (V, E)$ be an undirected graph. A vertex coloring of $G$ is an assignment of colors to the vertices of $G$ so that no two vertices with the same color are adjacent to each other in $G$. Equivalently, a vertex coloring of $G$ is a partition of $V$ into vertex subsets so that no two vertices in the same subset are adjacent to each other. A vertex coloring is also called just coloring of $G$. If $G$ has a coloring with $k$ colors, we say $G$ is $k$-colorable.
Let $G = (V, E)$ be an undirected graph.

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Graph Coloring Problem

Input: An undirected graph $G = (V, E)$
Output: Find a vertex coloring of $G$ using as few colors as possible.

Chromatic Number $\chi(G) =$ the smallest $k$ such that $G$ is $k$-colorable

$\chi(G) = 1$ iff $G$ has no edges.

$\chi(G) = 2$ iff $G$ is a bipartite graph with at least 1 edge.

Graph Coloring is a very hard problem. The problem can be solved in poly-time only for special graphs.
Scheduling All Intervals

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Four Color Theorem

Every planar graph can be colored using at most 4 colors.
Scheduling All Intervals

Four Color Theorem

Every planar graph can be colored using at most 4 colors.

$G$ is a planar graph if it can be drawn on the plane so that no two edges cross.

Both graphs (a) and (b) are planar graphs. The graph (a) has a 3-coloring. The graph (b) requires 4 colors, because all 4 vertices are adjacent to each other, and hence each vertex must have a different color.
Interval Graph

$G = (V, E)$ is called an interval graph if it can be represented as follows:

- Each vertex $p \in V$ represents an interval $[b_p, f_p)$.
- $(p, q) \in E$ if and only if the two intervals $I_p$ and $I_q$ overlap.
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Scheduling All Intervals

- It is easy to see that the problem of scheduling all intervals is precisely the graph coloring problem for interval graphs.

We discuss a greedy algorithm for solving this problem. It is not easy to prove the greedy choice property for this greedy strategy. We show the correctness of the algorithm by other methods. We use queues $Q_1, Q_2, ...$ to hold the subsets of intervals. (You can think that each $Q_i$ is a CPU, and if an interval $I_p = [b_p, f_p]$ is put into $Q_i$, the job $p$ is run on that CPU.)

Initially all queues are empty. When we consider an interval $[b_p, f_p]$ and a queue $Q_i$, we look at the last interval $[b_t, f_t]$ in $Q_i$. If $f_t \leq b_p$, we say $Q_i$ is available for $[b_p, f_p]$. (Meaning: the CPU $Q_i$ has finished the last job assigned to it. So it is ready to run the job $[b_p, f_p]$.)
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Scheduling All Intervals

Greedy-Schedule-All-Intervals

1. sort the intervals according to increasing $b_p$ value: $b_1 \leq b_2 \leq \cdots \leq b_n$
2. $k = 0$ (k will be the number of queues we need.)
3. for $p = 1$ to $n$ do:
   4. look at $Q_1, Q_2, \ldots Q_k$, put $[b_p, f_p)$ into the first available $Q_i$.
5. if no current queue is available:
   - increase $k$ by 1;
   - open a new empty queue;
   - put $[b_p, f_p)$ into this new queue.
6. output $k$ and $Q_1, \ldots, Q_k$
Scheduling All Intervals

I1 I2 I3
I4 I5 I6
I7 I8

After Sorting

Q1
Q2
Q3
Proof of correctness:

- We only put intervals into available queues. So each queue contains only non-overlapping intervals.
Scheduling All Intervals

Proof of correctness:

- We only put intervals into available queues. So each queue contains only non-overlapping intervals.
- We need to show the algorithm uses minimum number of queues. (Namely, partition intervals into minimum number of subsets.)
  - If the input contains $k$ mutually overlapping intervals, we must use at least $k$ queues. (Because no two such intervals can be placed into the same queue.)
Proof of correctness:

- We only put intervals into available queues. So each queue contains only non-overlapping intervals.

- We need to show the algorithm uses minimum number of queues. (Namely, partition intervals into minimum number of subsets.)

  - If the input contains $k$ mutually overlapping intervals, we must use at least $k$ queues. (Because no two such intervals can be placed into the same queue.)

  - When the algorithm opens a new empty queue $Q_k$ for an interval $[b_p, f_p)$, none of the current queues $Q_1, \cdots, Q_{k-1}$ is available. This means that the last intervals in $Q_1, \cdots, Q_{k-1}$ all overlap with $[b_p, f_p)$. Hence the input contains $k$ mutually overlapping intervals.
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  - The algorithm uses $k$ queues. By the observation above, this is the smallest possible.
Scheduling All Intervals

Runtime Analysis:

- Sorting takes $O(n \log n)$ time.
- The loop runs $n$ times.
- The loop body scans $Q_1, \ldots, Q_k$ to find the first available queue. So it takes $O(k)$ time.
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In the worst case, $k$ can be $\Theta(n)$. Hence, the worst case runtime is $\Theta(n^2)$. 