Outline

1. Insertion Sort

2. Divide & Conquer Strategy and Merge Sort

3. Master Theorem
Binary Search

- **Input:** Sorted array $A[1..n]$ and a number $x$
- **Output:** Find $i$ such that $A[i] = x$, if no such $i$ exists, output “no”.

We use a function `BinarySearch(A, p, r, x)` that searches $x$ in $A[p..r]$.

```
1: if p = r then
2: if A[p] = x return p
3: if A[p] ̸= x return “no”
4: else
5: q = (p + r) / 2
6: if A[q] = x return q
7: if A[q] > x call BinarySearch(A, p, q - 1, x)
8: if A[q] < x call BinarySearch(A, q + 1, r, x)
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\begin{align*}
\text{BinarySearch}(A, p, r, x) & \\
1: & \text{ if } p = r \text{ then} \\
2: & \quad \text{ if } A[p] = x \text{ return } p \\
3: & \quad \text{ if } A[p] \neq x \text{ return “no”} \\
4: & \text{ else} \\
5: & \quad q = (p + r)/2 \\
6: & \quad \text{ if } A[q] = x \text{ return } q \\
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9: & \text{ end if}
\end{align*}
Binary Search

Input: Sorted array $A[1..n]$ and a number $x$
Output: Find $i$ such that $A[i] = x$, if no such $i$ exists, output “no”.

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$\text{BinarySearch}(A, p, r, x)$

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8: if $A[q] < x$ call $\text{BinarySearch}(A, q + 1, r, x)$
9: end if
Insertion Sort

Sorting

- Input: An array \( A[1..n] \)
- Output: Re-ordered \( A \) such that \( A[i] \leq A[i+1] \) for any \( 1 \leq i \leq n - 1 \).
**Insertion Sort**

\texttt{InsertionSort}(\texttt{A[1..n]})

1: \textbf{For} \( i = 1 \) to \( n - 1 \)
2: \textbf{if} \( A[i] > A[i+1] \) \textbf{then}
3: \hspace{1em} \textbf{if} \( i = 1 \) \textbf{then}
4: \hspace{2em} \text{swap} \( A[i] \) and \( A[i+1] \)
5: \hspace{1em} \textbf{else}
6: \hspace{2em} \textbf{if} \( A[1] > A[i+1] \) \textbf{then}
7: \hspace{3em} \text{Insert} \( A[i+1] \) \text{ before} \( A[1] \)
8: \hspace{2em} \textbf{else}
9: \hspace{3em} \text{Find} \( A[i'] \) \text{ such that} \( 1 \leq i' < i, A[i'] \leq A[i+1] < A[i' + 1] \),
\hspace{3em} \text{insert} \( A[i+1] \) \text{ between} \( A[i'] \) and \( A[i' + 1] \)
10: \hspace{1em} \textbf{end if}
11: \hspace{1em} \textbf{end if}
12: \textbf{end if}
Analysis for Insertion Sort

At most $\theta \left( \sum_{i=1}^{n} i \right) = \theta(n^2)$ movements and comparisons $\Rightarrow T(n) = \theta(n^2)$.

Using binary search, we can reduce the number of comparisons to $\theta \left( \sum_{i=1}^{n} \log i \right) = \theta(n \log n)$.

Sometimes comparison is more expensive than movement.

Only $O(1)$ extra space.
Analysis for Insertion Sort

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  \( \longrightarrow T(n) = \theta(n^2). \)
Analysis for Insertion Sort

- At most $\theta\left(\sum_{i=1}^{n} i\right) = \theta(n^2)$ movements and comparisons
  $\implies T(n) = \theta(n^2)$.

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Analysis for Insertion Sort

- At most \( \theta(\sum_{i=1}^{n} i) = \theta(n^2) \) movements and comparisons \( \implies T(n) = \theta(n^2) \).
- Using binary search, we can reduce the number of comparisons to \( \theta(\sum_{i=1}^{n} \log i) = \theta(n \log n) \). Sometimes \textit{comparison} is more expensive than \textit{movement}.
- Only \( O(1) \) extra space.
Outline

1. Insertion Sort

2. Divide & Conquer Strategy and Merge Sort

3. Master Theorem
Divide and Conquer Strategy

- Algorithm design is more an art, less so a science.
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**Divide and Conquer**

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- Solve each subproblem (usually by recursive calls).
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For each of them, we will discuss a few examples, and try to identify common schemes.

Divide and Conquer

- Divide the problem into smaller subproblems (of the same type).
- Solve each subproblem (usually by recursive calls).
- Combine the solutions of the subproblems into the solution of the original problem.
Merge Sort

Input: an array $A[1..n]$
Output: Sort $A$ into increasing order.
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- Use a recursive function $\text{MergeSort}(A, p, r)$.
- In main program, we call $\text{MergeSort}(A, 1, n)$. 
Merge Sort

**MergeSort**\(A, p, r\)

1: if \(p < r\) then
2: \(q = (p + r)/2\)
3: MergeSort\(A, p, q\)
4: MergeSort\(A, q + 1, r\)
5: Merge\(A, p, q, r\)
6: else
7: do nothing
8: end if
Merge Sort

MergeSort(A, p, r)

1: if (p < r) then
2:    q = (p + r)/2
3:    MergeSort(A, p, q)
4:    MergeSort(A, q + 1, r)
5:    Merge(A, p, q, r)
6: else
7:    do nothing
8: end if

- Divide A[p..r] into two sub-arrays of equal size.
Merge Sort

\textbf{MergeSort}(A, p, r)

1: \textbf{if} \ (p < r) \ \textbf{then}
2: \hspace{1em} q = (p + r)/2
3: \hspace{1em} \text{MergeSort}(A, p, q)
4: \hspace{1em} \text{MergeSort}(A, q + 1, r)
5: \hspace{1em} \text{Merge}(A, p, q, r)
6: \textbf{else}
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- Divide $A[p..r]$ into two sub-arrays of equal size.
- Sort each sub-array by recursive call.
Merge Sort

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- Divide \(A[p..r]\) into two sub-arrays of equal size.
- Sort each sub-array by recursive call.
- **Merge**\((A, p, q, r)\) is a procedure that, assuming \(A[p..q]\) and \(A[q + 1..r]\) are sorted, merge them into sorted \(A[p..r]\)
- It can be done in \(\Theta(k)\) time where \(k = r - p\) is the number of elements to be sorted.
Analysis of MergeSort

Let $T(n)$ be the runtime function of MergeSort($A[1..n]$). Then:

$$T(n) = \begin{cases} 
O(1) & \text{if } n = 1 \\
2T(n/2) + \Theta(n) & \text{if } n > 1 
\end{cases}$$
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- Otherwise, we make 2 recursive calls. The input size of each is $n/2$. Hence the runtime $2T(n/2)$.
- $\Theta(n)$ is the time needed by $\text{Merge}(A, p, q, r)$ and all other processing.
Outline

1. Insertion Sort

2. Divide & Conquer Strategy and Merge Sort

3. Master Theorem
Master Theorem

For DaC algorithms, the runtime function often satisfies:

\[
T(n) = \begin{cases} 
O(1) & \text{if } n \leq n_0 \\
 aT(n/b) + \Theta(f(n)) & \text{if } n > n_0
\end{cases}
\]
Master Theorem

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\end{cases} \]

- If \( n \leq n_0 \) (\( n_0 \) is a small constant), we solve the problem directly without recursive calls. Since the input size is fixed (bounded by \( n_0 \)), it takes \( O(1) \) time.
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- \( \Theta(f(n)) \) is the time needed by all other processing.
- \( T(n) = ? \)
Master Theorem

Master Theorem (Theorem 4.1, Cormen’s book.)

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.

2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.

3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and $af(n/b) \leq cf(n)$ for some $c < 1$ for sufficiently large $n$, then $T(n) = \Theta(f(n))$.

Example: MergeSort

We have $a = 2, b = 2$, hence $\log_b a = \log_2 2 = 1$. So $f(n) = \Theta(n) = \Theta(n^{\log_b a})$.

By statement (2), $T(n) = \Theta(n \log n)$.
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Binary Search

- **Input:** Sorted array $A[1..n]$ and a number $x$
- **Output:** Find $i$ such that $A[i] = x$, if no such $i$ exists, output “no”.

```cpp
BinarySearch(A, p, r, x)
1: if p = r then
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Analysis of Binary Search

- If \( n = p - r + 1 = 1 \), it takes \( O(1) \) time.
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- If \( n = p - r + 1 = 1 \), it takes \( O(1) \) time.
- If not, we make at most one recursive call, with size \( n/2 \).
Analysis of Binary Search

- If $n = p - r + 1 = 1$, it takes $O(1)$ time.
- If not, we make at most one recursive call, with size $n/2$.
- All other processing take $f(n) = \Theta(1)$ time
Analysis of Binary Search

- If \( n = p - r + 1 = 1 \), it takes \( O(1) \) time.
- If not, we make at most one recursive call, with size \( n/2 \).
- All other processing take \( f(n) = \Theta(1) \) time.
- So \( a = 1, b = 2 \) and \( f(n) = \Theta(n^0) \) time.
  - Since \( \log_b a = \log_2 1 = 0 \), \( f(n) = \Theta(n^{\log_b a}) \).
Analysis of Binary Search

- If \( n = p - r + 1 = 1 \), it takes \( O(1) \) time.
- If not, we make at most one recursive call, with size \( n/2 \).
- All other processing take \( f(n) = \Theta(1) \) time
- So \( a = 1 \), \( b = 2 \) and \( f(n) = \Theta(n^0) \) time.
  Since \( \log_b a = \log_2 1 = 0 \), \( f(n) = \Theta(n^{\log_b a}) \).
- Hence \( T(n) = \Theta(n^{\log_b a \log n}) = \Theta(\log n) \).
A function makes 4 recursive calls, each with size \( n/2 \). Other processing takes \( f(n) = \Theta(n^3) \) time.
Example

A function makes 4 recursive calls, each with size \(n/2\). Other processing takes \(f(n) = \Theta(n^3)\) time.

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T(n) = 4T(n/2) + \Theta(n^3)
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We have \( a = 4, \ b = 2 \). So \( \log_b a = \log_2 4 = 2 \).
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We have \( a = 4, b = 2 \). So \( \log_b a = \log_2 4 = 2 \).

\[
f(n) = n^3 = \Theta(n^{\log_b a + 1}) = \Omega(n^{\log_b a + 0.5}).
\]

This is the case 3 of Master Theorem. We need to check the 2\textsuperscript{nd} condition:
A function makes 4 recursive calls, each with size \( n/2 \). Other processing takes \( f(n) = \Theta(n^3) \) time.

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This is the case 3 of Master Theorem. We need to check the 2\(^{nd} \) condition:

\[
a \cdot f(n/b) = 4 \left(\frac{n}{2}\right)^3 = \frac{4}{8}n^3 = \frac{1}{2} \cdot f(n)
\]

If we let \( c = 1/2 < 1 \), we have: \( a \cdot f(n/b) \leq c \cdot f(n) \).

Hence by case 3, \( T(n) = \Theta(f(n)) = \Theta(n^3) \).
Master Theorem

If $f(n)$ has the form $f(n) = \Theta(n^k)$ for some $k \geq 0$, We have the following:

A simpler version of Master Theorem

$$T(n) = \begin{cases} 
O(1) & \text{if } n \leq n_0 \\
aT(n/b) + \Theta(n^k) & \text{if } n > n_0
\end{cases}$$

1. If $k < \log_b a$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $k = \log_b a$, then $T(n) = \Theta(n^k \log n)$.
3. If $k > \log_b a$, then $T(n) = \Theta(n^k)$.
Master Theorem

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A simpler version of Master Theorem

$$T(n) = \begin{cases} 
O(1) & \text{if } n \leq n_0 \\
\alpha T(n/b) + \Theta(n^k) & \text{if } n > n_0
\end{cases}$$

1. If $k < \log_b a$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $k = \log_b a$, then $T(n) = \Theta(n^k \log n)$.
3. If $k > \log_b a$, then $T(n) = \Theta(n^k)$.

Only the case 3 is different. In this case, we need to check the 2nd condition. Because $k > \log_b a$, $b^k > a$ and $a/b^k < 1$:

$$a \cdot f(n/b) = a \cdot \left(\frac{n}{b}\right)^k = \frac{a}{b^k} \cdot f(n) = c \cdot f(n)$$

where $c = \frac{a}{b^k} < 1$, as needed.
How to understand/memorize Master Theorem?
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The cost of a DaC algorithm can be divided into two parts:
How to understand/memorize Master Theorem?

The cost of a DaC algorithm can be divided into two parts:

1. The total cost of all recursive calls is \( \Theta(n^{\log_b a}) \).

If (1) > (2), (1) dominates the total cost: \( T(n) = \Theta(n^{\log_b a}) \).

If (1) < (2), (2) dominates the total cost: \( T(n) = \Theta(f(n)) \).

If (1) = (2), the cost of two parts are about the same, somehow we have an extra factor \( \log n \).

The proof of Master Theorem is given in textbook.

We'll illustrate two examples in class.
Master Theorem

How to understand/memorize Master Theorem?

The cost of a DaC algorithm can be divided into two parts:

1. The total cost of all recursive calls is \( \Theta(n^{\log_b a}) \).
2. The total cost of all other processing is \( \Theta(f(n)) \).

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Theorem

If \( T(n) = aT(n/b) + f(n) \), where \( f(n) = \Theta(n^{\log_b a} (\log n)^k) \), then

\[ T(n) = \Theta(n^{\log_b a} (\log n)^{k+1}). \]
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$$T(n) = 2T(n/2) + \Theta(n \log n)$$

- $a = 2$, $b = 2$, $\log_a b = \log_2 2 = 1$.
- $f(n) = n^1 \log n = n^{\log_b a} \log n$
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In the above example, $T(n) = \Theta(n \log^2 n)$