Outline

1. Compare the growth rate of functions
2. Limit Test
3. L’Hospital Rule
4. Stirling Formula
Compare the growth rate of functions

We have two algorithms $A_1$ and $A_2$ for solving the same problem, with runtime functions $T_1(n)$ and $T_2(n)$, respectively. Which algorithm is more efficient?

We compare the growth rate of $T_1(n)$ and $T_2(n)$. If $T_1(n) = \Theta(T_2(n))$, then the efficiency of the two algorithms are about the same (when $n$ is large). If $T_1(n) = o(T_2(n))$, then the efficiency of the algorithm $A_1$ will be better than that of algorithm $A_2$ (when $n$ is large).

By using the definitions, we can directly show whether $T_1(n) = O(T_2(n))$, or $T_1(n) = \Omega(T_2(n))$. However, it is not easy to prove the relationship of two functions in this way.
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If $T_1(n) = \Theta(T_2(n))$, then the efficiency of the two algorithms are about the same (when $n$ is large).

If $T_1(n) = o(T_2(n))$, then the efficiency of the algorithm $A_1$ will be better than that of algorithm $A_2$ (when $n$ is large).

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Outline

1. Compare the growth rate of functions
2. Limit Test
3. L’Hospital Rule
4. Stirling Formula
Limit Test

Limit Test is a powerful method for comparing functions.

Let $T_1(n)$ and $T_2(n)$ be two functions. Let $c = \lim_{n \to \infty} \frac{T_1(n)}{T_2(n)}$.

1. If $c$ is a constant $> 0$, then $T_1(n) = \Theta(T_2(n))$.
2. If $c = 0$, then $T_1(n) = o(T_2(n))$.
3. If $c = \infty$, then $T_1(n) = \omega(T_2(n))$.
4. If $c$ does not exists (or if we do not know how to compute $c$), the limit test fails.
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Proof of (1): $c = \lim_{n \to \infty} \frac{T_1(n)}{T_2(n)}$ means: $\forall \epsilon > 0$, there exists $n_0 \geq 0$ such that for any $n \geq n_0$: $\left| \frac{T_1(n)}{T_2(n)} - c \right| \leq \epsilon$; or equivalently: $c - \epsilon \leq \frac{T_1(n)}{T_2(n)} \leq c + \epsilon$. Let $\epsilon = c/2$ and let $c_1 = c - \epsilon = c/2$ and $c_2 = c + \epsilon = 3c/2$, we have

$$c_1T_2(n) \leq T_1(n) \leq c_2T_2(n)$$

for all $n \geq n_0$. Thus $T_1(n) = \Theta(T_2(n))$ by definition.
Example

Example 1

\[ T_1(n) = 10n^2 + 15n - 60, \quad T_2(n) = n^2 \]

\[
\lim_{n \to \infty} \frac{T_1(n)}{T_2(n)} = \lim_{n \to \infty} \frac{10n^2 + 15n - 60}{n^2} = \lim_{n \to \infty} \left(10 + \frac{15}{n} - \frac{60}{n^2}\right) = 10 + 0 - 0 = 10
\]

Since 10 is a constant \(> 0\), we have \(T_1(n) = \Theta(T_2(n)) = \Theta(n^2)\) by the statement 1 of Limit Test (as expected).
The log functions are very useful in algorithm analysis.
The **log functions** are very useful in algorithm analysis.

\[
\begin{align*}
lg &= \log_2 n \\
\log n &= \log_{10} n \\
\ln n &= \log_e n
\end{align*}
\]

(\(\ln n\) is the log function with the **natural base** \(e = 2.71828 \ldots\)).
For any $1 < a, b$, \( \log_b n = \frac{\log_a n}{\log_a b} = \log_b a \cdot \log_a n \).

Proof: Let \( k = \log_b n \). By definition: \( n = b^k \).
For any $1 < a, b$, $\log_b n = \frac{\log_a n}{\log_a b} = \log_b a \cdot \log_a n$.

Proof: Let $k = \log_b n$. By definition: $n = b^k$.
Take $\log_a$ on both sides: $\log_a n = \log_a (b^k) = k \cdot \log_a b$
For any $1 < a, b$, \( \log_b n = \frac{\log_a n}{\log_a b} = \log_b a \cdot \log_a n \).

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Take \( \log_a \) on both sides: \( \log_a n = \log_a (b^k) = k \cdot \log_a b \)
This implies: \( \log_b n = k = \frac{\log_a n}{\log_a b} \).

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For any $1 < a, b$, \( \log_b n = \frac{\log_a n}{\log_a b} = \log_b a \cdot \log_a n \).

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Take \( \log_a \) on both sides: \( \log_a n = \log_a (b^k) = k \cdot \log_a b \)
This implies: \( \log_b n = k = \frac{\log_a n}{\log_a b} \).

Let \( n = a \) in this formula and note \( 1 = \log_a a \):
\[
\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}
\]
For any $1 < a, b$, $\log_b n = \frac{\log_a n}{\log_a b} = \log_b a \cdot \log_a n$.

Proof: Let $k = \log_b n$. By definition: $n = b^k$.
Take $\log_a$ on both sides: $\log_a n = \log_a (b^k) = k \cdot \log_a b$
This implies: $\log_b n = k = \frac{\log_a n}{\log_a b}$.

Let $n = a$ in this formula and note $1 = \log_a a$:

$$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}$$

This proves the second part of the formula.
Outline

1. Compare the growth rate of functions
2. Limit Test
3. L’Hospital Rule
4. Stirling Formula
L’Hospital Rule

If \( \lim_{n \to \infty} f(n) = 0 \) and \( \lim_{n \to \infty} g(n) = 0 \), then

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{f'(n)}{g'(n)}
\]
L’Hospital Rule

If \( \lim_{n \to \infty} f(n) = 0 \) and \( \lim_{n \to \infty} g(n) = 0 \), then

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\]

If \( \lim_{n \to \infty} f(n) = \infty \) and \( \lim_{n \to \infty} g(n) = \infty \), then

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{f'(n)}{g'(n)}
\]
Example

Example 2

\( T_1(n) = n^2 + 6, \ T_2(n) = n \lg n \). (Recall: \( \lg n = \log_2 n \).)

\[
\lim_{n \to \infty} \frac{T_1(n)}{T_2(n)} = \lim_{n \to \infty} \frac{n^2 + 6}{n \lg n} = \lim_{n \to \infty} \frac{n + \frac{6}{n}}{\lg n} \\
= \lim_{n \to \infty} \frac{1 - \frac{6}{n^2}}{\frac{1}{\ln 2 \cdot n}} \quad \text{(by L’Hospital Rule)} \\
= \ln 2 \lim_{n \to \infty} (n - \frac{6}{n}) = \ln 2 (\infty - 0) = \infty
\]

By Limit Test, we have \( n^2 + 6 = \omega(n \lg n) \).
Example 3

\[ T_1(n) = (\ln n)^k, \quad T_2(n) = n^\epsilon, \] where \( k > 0 \) is any (large) constant and \( \epsilon > 0 \) is any (small) constant. (Recall: \( \ln n = \log_e n \).

\[
\lim_{n \to \infty} \frac{T_1(n)}{T_2(n)} = \lim_{n \to \infty} \frac{(\ln n)^k}{n^\epsilon} \quad \text{(use L'Hospital Rule)}
\]

\[
= \lim_{n \to \infty} \frac{k(\ln n)^{k-1} \cdot (1/n)}{\epsilon n^{(\epsilon - 1)}}
\]

\[
= \frac{k}{\epsilon} \lim_{n \to \infty} \frac{(\ln n)^{k-1}}{n^{\epsilon}} \quad \text{(use L'Hospital Rule again and simplify)}
\]

\[
= \frac{k(k-1)}{\epsilon^2} \lim_{n \to \infty} \frac{(\ln n)^{k-2}}{n^{\epsilon}} \quad \text{(use L'Hospital Rule} \ k \text{ times)}
\]

\[
\ldots
\]

\[
= \frac{k(k-1) \cdots 2 \cdot 1}{\epsilon^k} \lim_{n \to \infty} \frac{1}{n^{\epsilon}} = 0
\]
Example

Example 3

\[ T_1(n) = (\ln n)^k, \quad T_2(n) = n^\epsilon, \] where \( k > 0 \) is any (large) constant and \( \epsilon > 0 \) is any (small) constant. (Recall: \( \ln n = \log_e n \).)

\[
\lim_{n \to \infty} \frac{T_1(n)}{T_2(n)} = \lim_{n \to \infty} \frac{(\ln n)^k}{n^\epsilon} \quad \text{(use L’Hospital Rule)}
\]

\[
= \lim_{n \to \infty} \frac{k(\ln n)^{k-1} \times (1/n)}{\epsilon n^{(\epsilon-1)}}
\]

\[
= \frac{k}{\epsilon} \lim_{n \to \infty} \frac{(\ln n)^{k-1}}{n^{\epsilon}} \quad \text{(use L’Hospital Rule again and simplify)}
\]

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= \frac{k(k-1)}{\epsilon^2} \lim_{n \to \infty} \frac{(\ln n)^{k-2}}{n^{\epsilon}} \quad \text{(use L’Hospital Rule \( k \) times)}
\]

\[
\vdots \]

\[
= \frac{k(k-1)\cdots2\cdot1}{\epsilon^k} \lim_{n \to \infty} \frac{1}{n^{\epsilon}} = 0
\]

So by Limit Test, \( (\ln n)^k = o(n^\epsilon) \) for any \( k \) and \( \epsilon \). For example, take \( k = 100 \) and \( \epsilon = 0.01 \), we have \( (\ln n)^{100} = o(n^{0.01}) \).
Example

Example 4

\( T_1(n) = n^k, \quad T_2(n) = a^n, \) where \( k > 0 \) is any (large) constant and \( a > 1 \) is any constant bigger than 1.

\[
\begin{align*}
\lim_{n \to \infty \frac{T_1(n)}{T_2(n)} &= \lim_{n \to \infty \frac{n^k}{a^n}} \quad \text{(using L’Hospital Rule)} \\
&= \lim_{n \to \infty \frac{k \cdot n^{k-1}}{\ln a \cdot a^n} \quad = \frac{k}{\ln a} \lim_{n \to \infty \frac{n^{k-1}}{a^n}} \quad \text{(using L’Hospital Rule } k \text{ times)} \\
&= \frac{k(k-1) \cdots 2 \cdot 1}{(\ln a)^k} \lim_{n \to \infty \frac{n^0}{a^n}} \\
&= \frac{k(k-1) \cdots 2 \cdot 1}{(\ln a)^k} \lim_{n \to \infty \frac{1}{a^n} = 0}
\end{align*}
\]
Example

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\( T_1(n) = n^k, \ T_2(n) = a^n, \) where \( k > 0 \) is any (large) constant and \( a > 1 \) is any constant bigger than 1.

\[
\lim_{n \to \infty} \frac{T_1(n)}{T_2(n)} = \lim_{n \to \infty} \frac{n^k}{a^n} \quad \text{(using L’Hospital Rule)}
\]

\[
= \lim_{n \to \infty} \frac{k \cdot n^{k-1}}{\ln a \cdot a^n} = \frac{k}{\ln a} \lim_{n \to \infty} \frac{n^{k-1}}{a^n} \quad \text{(using L’Hospital Rule \( k \) times)}
\]

\[
= \frac{k(k-1) \cdots 2 \cdot 1}{(\ln a)^k} \lim_{n \to \infty} \frac{n^0}{a^n}
\]

\[
= \frac{k(k-1) \cdots 2 \cdot 1}{(\ln a)^k} \lim_{n \to \infty} \frac{1}{a^n} = 0
\]

So by Limit Test, \( n^k = o(a^n) \) for any \( k > 0 \) and \( a > 1 \). For example, take \( k = 1000 \) and \( a = 1.001 \), we have \( n^{1000} = o((1.001)^n) \).
Example 5

$T_1(n) = \log_a n$, $T_2(n) = \log_b n$, where $a > 1$ and $b > 1$ are any two constants bigger than 1.

By the Log Base Change Formula: $\log_b n = \log_b a \cdot \log_a n$
By the Log Base Change Formula: \( \log_b n = \log_b a \cdot \log_a n \)

Thus: \( \lim_{n \to \infty} \frac{T_1(n)}{T_2(n)} = \lim_{n \to \infty} \frac{\log_a n}{\log_b n} = \lim_{n \to \infty} \frac{\log_a n}{\log_b a \cdot \log_a n} = \frac{1}{\log_b a} \)
Example 5

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By the Log Base Change Formula: \( \log_b n = \log_b a \cdot \log_a n \)

Thus: \( \lim_{n \to \infty} \frac{T_1(n)}{T_2(n)} = \lim_{n \to \infty} \frac{\log_a n}{\log_b n} = \lim_{n \to \infty} \frac{\log_a n}{\log_b a \cdot \log_a n} = \frac{1}{\log_b a} \)

Since \( \frac{1}{\log_b a} > 0 \) is a constant, we have \( \log_a n = \Theta(\log_b n) \) by Limit Test.
Example 5

Let $T_1(n) = \log_a n$, $T_2(n) = \log_b n$, where $a > 1$ and $b > 1$ are any two constants bigger than 1.

By the Log Base Change Formula: $\log_b n = \frac{\log_a n}{\log_a b}$

Thus: $\lim_{n \to \infty} \frac{T_1(n)}{T_2(n)} = \lim_{n \to \infty} \frac{\log_a n}{\log_b n} = \lim_{n \to \infty} \frac{\log_a n}{\log_b a \cdot \log_a n} = \frac{1}{\log_b a}$

Since $\frac{1}{\log_b a} > 0$ is a constant, we have $\log_a n = \Theta(\log_b n)$ by Limit Test.

So: the growth rates of the $\log$ functions are the same for any base $> 1$. 
Example 6

$T_1(n) = a^n$, $T_2(n) = b^n$, where $1 < a < b$ are any two constants.

We have: $\lim_{n \to \infty} \frac{T_1(n)}{T_2(n)} = \lim_{n \to \infty} \frac{a^n}{b^n} = \lim_{n \to \infty} (\frac{a}{b})^n = 0$. 

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Example

Example 6

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Thus: \( a^n = o\left(b^n\right) \) (for any \( 1 < a < b \)) by Limit Test.
Example 6

\[ T_1(n) = a^n, \ T_2(n) = b^n, \text{ where } 1 < a < b \text{ are any two constants.} \]

We have: \( \lim_{n \to \infty} \frac{T_1(n)}{T_2(n)} = \lim_{n \to \infty} \frac{a^n}{b^n} = \lim_{n \to \infty} (\frac{a}{b})^n = 0. \)

Thus: \( a^n = o(b^n) \) (for any \( 1 < a < b \)) by Limit Test.

The list of common functions:

The following list shows the functions commonly used in algorithm analysis, in the order of increasing growth rate (\( a, b, c, d, k, \epsilon \) are positive constants, \( \epsilon < 1, k > 1, d > 1 \) and \( a < b \)): 

- \( c \)
- \( \log_d n \)
- \( (\log_d n)^k \)
- \( n^\epsilon \)
- \( n \)
- \( n^k \)
- \( a^n \)
- \( b^n \)
- \( n! \)
- \( n^n \)
Example

Example 6

\[ T_1(n) = a^n, \ T_2(n) = b^n, \text{ where } 1 < a < b \text{ are any two constants.} \]

We have: \( \lim_{n \to \infty} \frac{T_1(n)}{T_2(n)} = \lim_{n \to \infty} \frac{a^n}{b^n} = \lim_{n \to \infty} (\frac{a}{b})^n = 0. \)
Thus: \( a^n = o(b^n) \) (for any \( 1 < a < b \)) by Limit Test.

The list of common functions:

The following list shows the functions commonly used in algorithm analysis, in the order of increasing growth rate \((a, b, c, d, k, \epsilon \text{ are positive constants, } \epsilon < 1, k > 1, d > 1 \text{ and } a < b)\):

\[ c, \log_d n, \ (\log_d n)^k, n^\epsilon, n, n^k, a^n, b^n, n!, n^n \]

in the sense that if \( f(n) \) and \( g(n) \) are any two consecutive functions in the list, we have \( f(n) = o(g(n)) \).
Example 7

\[ T_1(n) = n! \text{ and } T_2(n) = a^n (a > 1) \]
Example 7

\( T_1(n) = n! \) and \( T_2(n) = a^n \) (\( a > 1 \))

\[ \lim_{n \to \infty} \frac{a^n}{n!} = ? \]
Example 7

\( T_1(n) = n! \) and \( T_2(n) = a^n \) (\( a > 1 \))

1. \( \lim_{n \to \infty} \frac{a^n}{n!} = ? \)
2. L'Hospital Rule doesn't help: We don't know how to take derivative of \( n! \).
Example 7

$T_1(n) = n!$ and $T_2(n) = a^n \ (a > 1)$

- $\lim_{n \to \infty} \frac{a^n}{n!} = ?$
- L'Hôpital Rule doesn't help: We don't know how to take derivative of $n!$

\[
\frac{a^n}{n!} = \frac{a}{1} \cdot \frac{a}{2} \cdot \ldots \cdot \frac{a}{2[k]} \cdot \frac{a}{2[k] + 1} \cdot \ldots \cdot \frac{a}{n} \\
\underbrace{\frac{a}{2[k]} \cdot \ldots \cdot \frac{a}{n}}_{(n-2[k]) \text{ terms}} \\
\underbrace{\frac{a}{1} \cdot \frac{a}{2} \cdot \ldots \cdot \frac{a}{2[k]}}_{2[k] \text{ terms}}
\]
Example 7

\[ T_1(n) = n! \quad \text{and} \quad T_2(n) = a^n \quad (a > 1) \]

\[ \lim_{n \to \infty} \frac{a^n}{n!} = ? \]

L'Hospital Rule doesn't help: We don't know how to take derivative of \( n! \)

\[
\frac{a^n}{n!} = \frac{a}{1} \cdot \frac{a}{2} \cdot \frac{a}{2[a]} \cdot \frac{a}{2[a] + 1} \cdots \frac{a}{n}
\]

\[ 2[a] \text{ terms} \quad (n-2[a]) \text{ terms} \]

The first part is a constant \( c > 0 \). In the second part, each term < 1/2. So:
Examples

Example 7

\[ T_1(n) = n! \quad \text{and} \quad T_2(n) = a^n \quad (a > 1) \]

- \( \lim_{n \to \infty} \frac{a^n}{n!} = ? \)
- L’Hospital Rule doesn’t help: We don’t know how to take derivative of \( n! \)

\[
\frac{a^n}{n!} = \frac{a}{1} \cdot \frac{a}{2} \cdot \frac{a}{2[a]} \cdot \frac{a}{2[a] + 1} \cdots \frac{a}{n}
\]

\( 2[a] \) terms \quad \( (n-2[a]) \) terms

The first part is a constant \( c > 0 \). In the second part, each term < 1/2. So:

\[
0 \leq \lim_{n \to \infty} \frac{a^n}{n!} \leq c \cdot \lim_{n \to \infty} \left( \frac{1}{2} \right)^{(n-2[a])} = 0
\]
Example 7

\(T_1(n) = n! \quad \text{and} \quad T_2(n) = a^n \quad (a > 1)\)

- \(\lim_{n \to \infty} \frac{a^n}{n!} = ?\)
- L’Hospital Rule doesn’t help: We don’t know how to take derivative of \(n!\)

\[
a^n \cdot \frac{1}{n!} = \frac{a}{1} \cdot \frac{a}{2} \cdot \frac{a}{2[\alpha]} \cdot \frac{a}{2[\alpha] + 1} \cdots \frac{a}{n}
\]

\[
\text{2[\alpha] terms} \quad \text{2[\alpha] terms}
\]

The first part is a constant \(c > 0\). In the second part, each term < 1/2. So:

\[
0 \leq \lim_{n \to \infty} \frac{a^n}{n!} \leq c \cdot \lim_{n \to \infty} \left(\frac{1}{2}\right)^{(n - 2[\alpha])} = 0
\]

By Limit Test: \(a^n = o(n!)\).
1. Compare the growth rate of functions
2. Limit Test
3. L’Hospital Rule
4. Stirling Formula
Stirling Formula

\[ \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} \leq n! \leq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n}} \]

or:

\[ n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta \left(\frac{1}{n}\right)\right) \]

When \( n = 10; \)

- \( n! = 3628800. \)
- \( \sqrt{2\pi n} \left(\frac{n}{e}\right)^n = 3598696, \) 99% accurate.
Example 7 (another solution)

\( T_1(n) = n! \) and \( T_2(n) = a^n (a > 1) \)

\[
\lim_{n \to \infty} \frac{n!}{a^n} \geq \lim_{n \to \infty} \sqrt{2\pi n} \left( \frac{n}{ae} \right)^n = \lim_{n \to \infty} \sqrt{2\pi n} \cdot \lim_{n \to \infty} \left( \frac{n}{ae} \right)^n
\]
Example 7 (another solution)

\[ T_1(n) = n! \quad \text{and} \quad T_2(n) = a^n \quad (a > 1) \]

\[
\lim_{n \to \infty} \frac{n!}{a^n} \geq \lim_{n \to \infty} \sqrt{2\pi n} \left( \frac{n}{ae} \right)^n = \lim_{n \to \infty} \sqrt{2\pi n} \cdot \lim_{n \to \infty} \left( \frac{n}{ae} \right)^n
\]

The first limit is \( \infty \). The second limit goes to \( \infty \infty \). So it’s also \( \infty \). Thus \( \lim_{n \to \infty} \frac{n!}{a^n} = \infty \) and \( n! = \omega(a^n) \) by limit test.
Example 7 (another solution)

$T_1(n) = n!$ and $T_2(n) = a^n \; (a > 1)$

$$\lim_{n \to \infty} \frac{n!}{a^n} \geq \lim_{n \to \infty} \sqrt{2\pi n} \left(\frac{n}{ae}\right)^n = \lim_{n \to \infty} \sqrt{2\pi n} \cdot \lim_{n \to \infty} \left(\frac{n}{ae}\right)^n$$

The first limit is $\infty$. The second limit goes to $\infty \infty$. So it’s also $\infty$. Thus $\lim_{n \to \infty} \frac{n!}{a^n} = \infty$ and $n! = \omega(a^n)$ by limit test.

Example 8

$T_1(n) = n^n$ and $T_2(n) = n!$

By using similar method as in Example 7, we can show: $n! = o(n^n)$