1. (5 + 5 = 10 points) Compute the expression for $T(n)$:

1. $T(n) = 3T(n/4) + \sqrt{n}$;
2. $T(n) = 4T(n/3) + n^{1/3}$.

**Solution:** (1) $a = 3$ and $b = 4$, $\log_b a > 1/2$. We can just set $\epsilon = (\log_4 3 - 1/2)/2 > 0$, $n_0 = 10$, so $T(n) = \theta(n^{\log_4 3})$ by Master Theorem. (2) Similar to (1), we set $\epsilon = (\log_3 4 - 1/3)/2 > 0$, $n_0 = 10$, and $T(n) = \theta(n^{\log_3 4})$.

2. (8 points) Prove $\sum_{i=1}^{n} i \log i = \theta(n^2 \log n)$.

**Solution:** First, it is easy to know that $i \log i \leq n \log n$ for any $i = 1, \cdots, n$, so $\sum_{i=1}^{n} i \log i = O(n^2 \log n)$. Second,

$$\sum_{i=1}^{n} i \log i > \sum_{i=n/2}^{n} i \log i > \frac{n}{2} \times \frac{n}{2} \log \frac{n}{2},$$

Which implies that $\sum_{i=1}^{n} i \log i = \Omega(n^2 \log n)$. In total, $\sum_{i=1}^{n} i \log i = \theta(n^2 \log n)$.

3. (10 + 10 = 20 points) Let an array $A = [7, 10, 11, 2, 3, 0, 12]$. Please build (1) MinHeap and (2) Max-Heap for $A$.

**Solution:** (1) MinHeap

- $[7, 10, 11, 2, 3, 0, 12]$
- $[7, 10, 0, 2, 3, 11, 12]$
- $[7, 2, 0, 10, 3, 11, 12]$
- $[0, 2, 7, 10, 3, 11, 12]$

(2) MaxHeap

- $[7, 10, 11, 2, 3, 0, 12]$
- $[7, 10, 12, 2, 3, 0, 11]$
- $[7, 10, 12, 2, 3, 0, 11]$
- $[12, 10, 7, 2, 3, 0, 11]$
- $[12, 10, 11, 2, 3, 0, 7]$

4. (10 points) Is the following tree (Fig. 1) an AVL-tree? If not, please convert it to an AVL-tree.

**Solution:** No. The AVL-tree after conversion is shown in Fig. 2.
5. (8 + 8 = 16 points) Suppose we want to insert \{1, 5, 8, 12, 19, 15, 22\} into a chaining hash table with size \(m = 7\). Please show the final hash tables by the following two hash functions separately:

1. \(h(x) = x \mod 7\);
2. \(h(x) = ((2x + 5) \mod 13) \mod 7\).

**Solution:** See Fig. 3a for (1) and Fig. 3b for (2).

6. (8 points) Suppose Facebook stores a huge undirected graph for the social network, where each vertex represents a user, and if two users are friends with each other, they are connected by an edge. We assume that there are about \(10^9\) users, and each user has about 200 friends on average. Should we use adjacency list or adjacency matrix to represent this graph? Why?

**Solution:** Here, \(n = 10^9\) and \(m = \frac{1}{7} \times 200 \times n = 100 \times 10^9\). We can consider 100 as a constant comparing to \(10^9\). The space complexity is \(\Theta(n^2)\) for adjacency matrix and \(100 \times n \times \Theta(1) = \Theta(n)\) for adjacency list. So adjacency list saves quite a lot in space and we should use adjacency list.
7. (8 + 10 = 18 points) (1) Show the adjacency list of the above directed graph (Fig. 4), (2) then based on your adjacency list, show the induced spanning tree by DFS.

Solution: See Fig. 5a for (1) and Fig. 5b for (2).

8. (10 points) Use the same graph Fig. 4. Suppose each vertex represents a task, and each directed edge indicates a prerequisite. For example, the edge $v_3 \rightarrow v_4$ indicates that we can start task $v_4$ only when $v_3$ has been finished. Please use topological sort to schedule these 6 tasks.

Solution: See Fig. 6.