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# FUNDAMENTALS PHYSICS

## And Differential Equations

CSE 872 FALL 2009

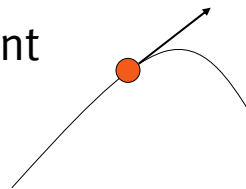
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# DYNAMICS

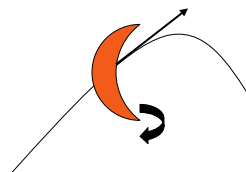
## Dynamics of a material point

*Ideal case, but often sufficient*



## Dynamics of a solid

*Including rotation, torques...*



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2

## POSITION, VELOCITY, ACCELERATION

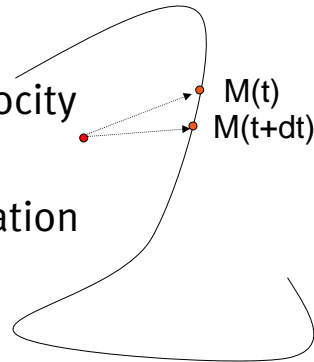
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$$\mathbf{V} = \lim_{dt \rightarrow 0} \frac{\mathbf{OM}(t+dt) - \mathbf{OM}(t)}{dt}$$

$$\mathbf{V} = d\mathbf{OM}/dt = \dot{\mathbf{O}\dot{\mathbf{M}}} = \text{velocity}$$

$$\|\mathbf{V}\| = \text{speed}$$

$$\mathbf{A} = d\mathbf{V}/dt = \dot{\mathbf{V}} = \text{acceleration}$$
$$= d^2\mathbf{OM}/dt^2 = \ddot{\mathbf{O}\dot{\mathbf{M}}}$$



## NEWTON LAW (MATERIAL POINT)

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$$\mathbf{F} = m \mathbf{A}$$

**F**: sum of the forces exerted

**m**: mass of the material point

## EXERCISE: MATERIAL POINT FALLING

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$$A(t) = g$$

$$V = V_0 + g t$$

$$M = M_0 + V_0 t + .5 * g t^2$$

Now, how can we find the motion of this point when some external forces are present?

## TEMPORAL INTEGRATION

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In the previous example, we were able to explicitly integrate the motion.

However, we must often numerically integrate the motion – For instance, Newton method:

$$A(t) = F(t) / m$$

$$V(t) = V(t-dt) + A(t) dt$$

$$M(t) = M(t-dt) + V(t) dt$$

Prb: accuracy, stability...

## POINT VS. OBJECT

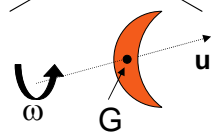
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We overlooked rotations! A real object can also spin on itself during motion.

Notion of **angular velocity**:

$$\mathbf{\omega} = \omega \mathbf{u}$$

angular speed    axis of rotation



(G: center of mass)

## ADDITIONAL LAW

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$$\mathbf{F} = m \mathbf{A}$$

$$\mathcal{M} = (\mathbf{J} \dot{\boldsymbol{\omega}})$$

$\mathcal{M}$ : External torques

$\mathbf{J}$ : Inertia matrix

$\mathbf{J} \dot{\boldsymbol{\omega}}$ : Cinematic momentum

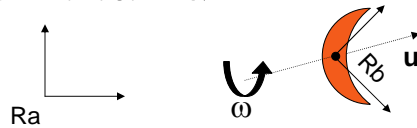
## WARNING !!

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Be extremely careful:

$$\dot{\mathbf{U}}_{/Ra} = \dot{\mathbf{U}}_{/Rb} + \boldsymbol{\omega} \times \mathbf{U}_{/Ra}$$

where  $\boldsymbol{\omega}$  is the angular velocity of Rb w.r.t. Ra



## INTEGRATION

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Just a bit more complex...

Quaternions make it easier

Topics for rigid body simulation

## OTHER THINGS TO KNOW

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Action/Reaction Principle:

$$\mathbf{F}_{1/2} = -\mathbf{F}_{2/1}$$

## DIFFERENTIAL EQUATIONS

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Integration is often needed

- Physics-based techniques use differential equations
- Either simply ODEs [we will review that again!]
- Or more complicated PDEs

# ODE

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## Ordinary Differential Equation

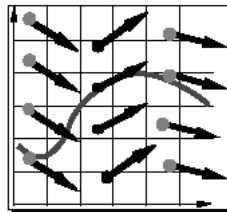
- Ex:  $dx/dt = v$
- Ex:  $dv/dt = a$
- Functions of only one variable
- Often, set of ODEs to solve
  - see Rigid Body
  - see all the physics-based models

# INITIAL VALUE PROBLEM

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A typical example:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t)) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$



$\mathbf{f}$  is like a “current”, driving the evolution of  $\mathbf{x}$

Sometimes, we can find a symbolic solution

- $\dot{\mathbf{x}} = -k \mathbf{x}$  (you have 10 seconds to solve this one)

Unfortunately,  $\mathbf{f}$  may be changed all the time...

# NUMERICAL INTEGRATION

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Discrete approximation of the real solution

Various techniques:

- Some ways are better than others
- Never perfect
- Always better if we know the type of solutions we should get...

# EULER'S METHOD

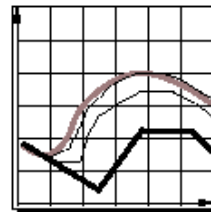
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Simplest method (first-order):

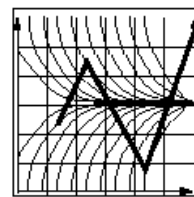
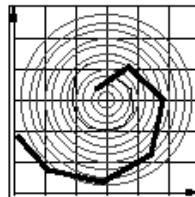
$$\mathbf{x}(t+dt) = \mathbf{x}(t) + \dot{\mathbf{x}}(t) dt + O(dt^2)$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t))$$

$$\Rightarrow \mathbf{x}(t+dt) \approx \mathbf{x}(t) + \mathbf{f}(\mathbf{x}(t)) dt$$



Can be really bad:



# MIDPOINT METHOD I

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Can we go higher?

$$x(t+dt) = x(t) + \dot{x}(t) dt + \frac{1}{2} \ddot{x}(t) dt^2 + O(dt^3)$$

$$\text{Chain rule: } \ddot{x}(t) = f'(x(t)) \dot{x}(t)$$

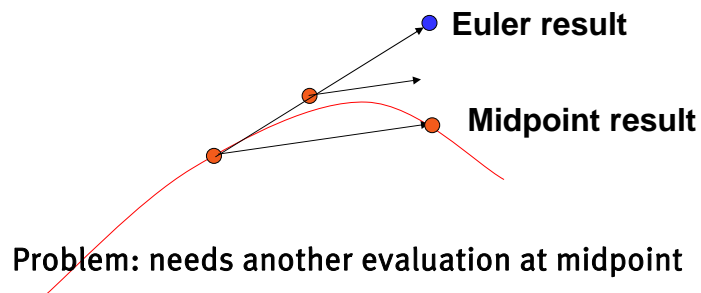
$$\text{Now: } f(x+dx) = f(x) + f'(x) dx + O(dx^2)$$

$$\text{Therefore: } x(t+dt) = x(t) + dt f(x(t) + \frac{1}{2} f(x))$$

# MIDPOINT METHOD II

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Geometric interpretation:



## OTHER “EXPLICIT” METHODS

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Midpoint = RK2 (Runge-Kutta 2<sup>nd</sup> order)  
RK4 is recommended when feasible

Adaptive time step:

- For Euler, we can define:  
 $e = |x_a - x_b|$  ( $x_a$ : one step,  $x_b$ : two steps)
- According to  $e$ , change  $dt$
- Can save a lot of computations

## PDE

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### Partial Derivative Equations

- For multivariate functions
- Ex: for a field  $u(x,t)$  ( $u$  = pressure, temperature, ...)

$$\frac{d^2u}{dt^2} = -v^2 \frac{d^2u}{dx^2} \quad (\text{wave propagation})$$

- E.g., fluid simulation (water, smoke, etc..)

## NUMERICAL INTEGRATION I

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Let's take:  $\frac{\partial u}{\partial t} = -v \frac{\partial u}{\partial x}$

Discretization of x and t:

- $x_j = x_0 + j \Delta x$
- $t_n = t_0 + n \Delta t$
- $u_j^n = u(x_j, t_n)$

Finite Differences:

$$\frac{\partial u}{\partial t} \Big|_{j,n} = \frac{u_j^{n+1} - u_j^n}{\Delta t} \quad \frac{\partial u}{\partial x} \Big|_{j,n} = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

## NUMERICAL INTEGRATION II

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We find:  $u_j^{n+1} = u_j^n - \frac{v\Delta t}{2\Delta x} (u_{j+1}^n - u_{j-1}^n)$

Problem: not stable at all

Why???

## STABILITY ANALYSIS

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Von Neumann method:

Consider:  $u_j^n = \xi(k)^n e^{ikj\Delta x}$

What does  $\xi$  become after one integration step?

Previous case:  $\xi(k) = 1 - i \frac{v\Delta t}{\Delta x} \sin(k\Delta x)$

Therefore,  $|\xi| \rightarrow 1 \Rightarrow$  **unconditionally unstable**

## NEW NUMERICAL SCHEME

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Slight change (Lax):

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{v\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n)$$

Now, stable if  $\frac{|v|\Delta t}{\Delta x} < 1$  !

- Reason:  $\xi(k) = \cos(k\Delta x) - i \frac{v\Delta t}{\Delta x} \sin(k\Delta x)$
- Stability condition: called **Courant condition**
- **Very general for PDE**
- Idea: never skip a point or you ask for trouble

## WHY DID IT WORK?

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Rewrite Lax method, you'll get:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{1}{2} \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta t}$$

So the "real" PDE we integrate is:

$$\frac{\partial u}{\partial t} = -v \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2\Delta t} \frac{\partial^2 u}{\partial x^2}$$

- Second-order term: diffusion, killing high frequencies
- Called **numerical viscosity**
- Good, but bad...

## HIGHER LEVEL OF KNOWLEDGE

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For the previous equation, we know that it's a propagation

Go with the wind!

$$\left. \frac{\partial u}{\partial x} \right|_{j,n} = \frac{u_{j+1}^n - u_j^n}{\Delta x}$$

Called: **upwind method** (also stable)

# SOMETHING IMPORTANT NOT COVERED

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**Implicit integration:**  $x(t+dt) \approx x(t) + f(x(t+dt)) dt$

Need caution

To prevent these cases:

