
Fundamentals Physics

And Differential Equations

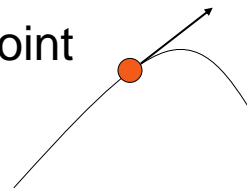
CSE 872 Fall 2011

1

Dynamics

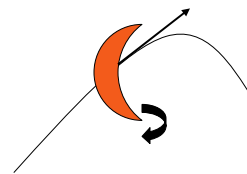
Dynamics of a material point

Ideal case, but often sufficient



Dynamics of a solid

Including rotation, torques...



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2

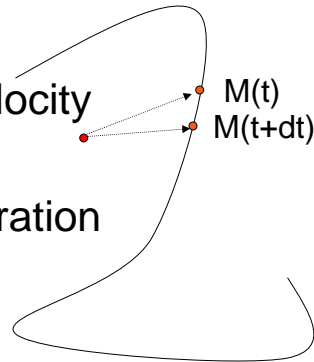
Position, Velocity, Acceleration

$$\mathbf{V} = \lim_{dt \rightarrow 0} \frac{\mathbf{OM}(t+dt) - \mathbf{OM}(t)}{dt}$$

$$\mathbf{V} = d\mathbf{OM}/dt = \dot{\mathbf{O}}\mathbf{M} = \text{velocity}$$

$$\|\mathbf{V}\| = \text{speed}$$

$$\mathbf{A} = d\mathbf{V}/dt = \dot{\mathbf{V}} = \text{acceleration}$$
$$= d^2\mathbf{OM}/dt^2 = \ddot{\mathbf{O}}\mathbf{M}$$



Newton Law (*material point*)

$$\mathbf{F} = m \mathbf{A}$$

F: sum of the forces exerted

m: mass of the material point

Exercise: Material point falling

$$\mathbf{A}(t) = \mathbf{g}$$

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{g} t$$

$$\mathbf{M} = \mathbf{M}_0 + \mathbf{V}_0 t + .5 * \mathbf{g} t^2$$

Now, how can we find the motion of this point when some external forces are present?

Temporal integration

In the previous example, we were able to explicitly integrate the motion.

However, we must often numerically integrate the motion – For instance, Newton method:

$$\mathbf{A}(t) = \mathbf{F}(t) / m$$

$$\mathbf{V}(t) = \mathbf{V}(t-dt) + \mathbf{A}(t) dt$$

$$\mathbf{M}(t) = \mathbf{M}(t-dt) + \mathbf{V}(t) dt$$

Prb: accuracy, stability...

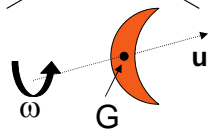
Point vs. Object

We overlooked rotations! A real object can also spin on itself during motion.

Notion of **angular velocity**:

$$\mathbf{v} = \boldsymbol{\omega} \mathbf{u}$$

angular speed axis of rotation



(G: center of mass)

Additional Law

$$\mathbf{F} = m \mathbf{A}$$

$$\mathcal{M} = (\mathbf{J} \dot{\boldsymbol{\omega}})$$

\mathcal{M} : External torques

\mathbf{J} : Inertia matrix

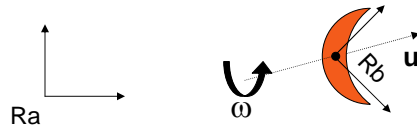
$\mathbf{J} \dot{\boldsymbol{\omega}}$: Cinematic momentum

Warning!!

Be extremely careful:

$$\dot{\mathbf{U}}_{/Ra} = \dot{\mathbf{U}}_{/Rb} + \boldsymbol{\omega} \times \mathbf{U}_{/Ra}$$

where $\boldsymbol{\omega}$ is the angular velocity of Rb w.r.t. Ra



Integration

Just a bit more complex...

Quaternions make it easier

Topics for rigid body simulation

Other things to know

Action/Reaction Principle:

$$\mathbf{F}_{1/2} = -\mathbf{F}_{2/1}$$

Differential Equations

Integration is often needed

- Physics-based techniques use differential equations
- Either simply ODEs [we will review that again!]
- Or more complicated PDEs

ODE

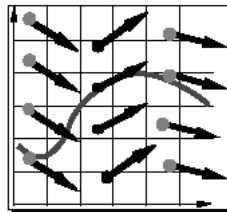
Ordinary Differential Equation

- Ex: $dx/dt = v$
- Ex: $dv/dt = a$
- Functions of only one variable
- Often, set of ODEs to solve
 - see Rigid Body
 - see all the physics-based models

Initial Value Problem

A typical example:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t)) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$



\mathbf{f} is like a “current”, driving the evolution of \mathbf{x}

Sometimes, we can find a symbolic solution

- $\dot{\mathbf{x}} = -k \mathbf{x}$ (you have 10 seconds to solve this one)

Unfortunately, \mathbf{f} may be changed all the time...

Numerical Integration

Discrete approximation of the real solution

Various techniques:

- Some ways are better than others
- Never perfect
- Always better if we know the type of solutions we should get...

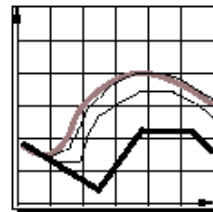
Euler's Method

Simplest method (first-order):

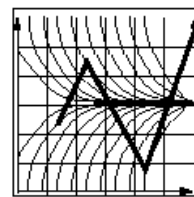
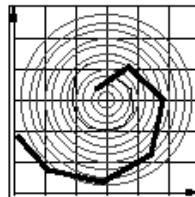
$$\mathbf{x}(t+dt) = \mathbf{x}(t) + \dot{\mathbf{x}}(t) dt + O(dt^2)$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t))$$

$$\Rightarrow \mathbf{x}(t+dt) \approx \mathbf{x}(t) + \mathbf{f}(\mathbf{x}(t)) dt$$



Can be really bad:



Midpoint Method I

Can we go higher?

$$\mathbf{x}(t+dt) = \mathbf{x}(t) + \dot{\mathbf{x}}(t) dt + \frac{1}{2} \ddot{\mathbf{x}}(t) dt^2 + O(dt^3)$$

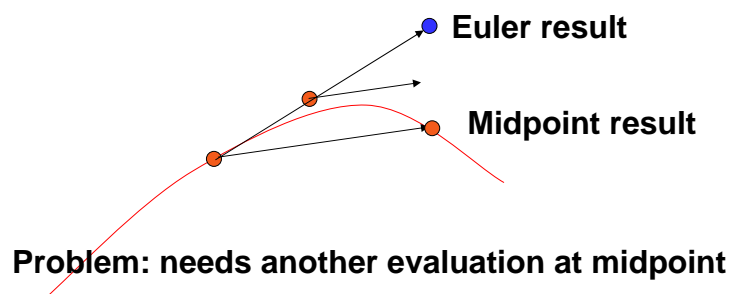
$$\text{Chain rule: } \ddot{\mathbf{x}}(t) = f(\mathbf{x}(t)) \dot{f}(\mathbf{x}(t))$$

$$\text{Now: } f(\mathbf{x}+d\mathbf{x}) = f(\mathbf{x}) + \dot{f}(\mathbf{x}) d\mathbf{x} + O(d\mathbf{x}^2)$$

$$\text{Therefore: } \mathbf{x}(t+dt) = \mathbf{x}(t) + dt f(\mathbf{x}+dt/2 f(\mathbf{x}))$$

Midpoint method II

Geometric interpretation:



Other “Explicit” Methods

Midpoint = RK2 (Runge-Kutta 2nd order)
RK4 is recommended when feasible

Adaptive time step:

- For Euler, we can define:
 $e = |x_a - x_b|$ (x_a : one step, x_b : two steps)
- According to e , change dt
- Can save a lot of computations

PDE

Partial Derivative Equations

- For multivariate functions
- Ex: for a field $u(x,t)$ (u = pressure, temperature,)

$$\frac{d^2u}{dt^2} = -v^2 \frac{d^2u}{dx^2} \quad (\text{wave propagation})$$

- E.g., fluid simulation (water, smoke, etc..)

Numerical Integration I

Let's take: $\frac{\partial u}{\partial t} = -v \frac{\partial u}{\partial x}$

Discretization of x and t :

- $x_j = x_0 + j \Delta x$
- $t_n = t_0 + n \Delta t$
- $u_j^n = u(x_j, t_n)$

Finite Differences:

$$\left. \frac{\partial u}{\partial t} \right|_{j,n} = \frac{u_j^{n+1} - u_j^n}{\Delta t} \quad \left. \frac{\partial u}{\partial x} \right|_{j,n} = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

Numerical Integration II

We find: $u_j^{n+1} = u_j^n - \frac{v\Delta t}{2\Delta x} (u_{j+1}^n - u_{j-1}^n)$

Problem: not stable at all

Why???

Stability analysis

Von Neumann method:

$$\text{Consider: } u_j^n = \xi(k)^n e^{ikj\Delta x}$$

What does ξ become after one integration step?

$$\text{Previous case: } \xi(k) = 1 - i \frac{v\Delta t}{\Delta x} \sin(k\Delta x)$$

Therefore, $|\xi| > 1 \Rightarrow$ **unconditionally instable**

New numerical scheme

Slight change (Lax):

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{v\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n)$$

Now, stable if $\frac{|v|\Delta t}{\Delta x} < 1$!

- Reason: $\xi(k) = \cos(k\Delta x) - i \frac{v\Delta t}{\Delta x} \sin(k\Delta x)$
- Stability condition: called **Courant condition**
- **Very general for PDE**
- Idea: never skip a point or you ask for trouble

Why did it work?

Rewrite Lax method, you'll get:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{1}{2} \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta t}$$

So the “real” PDE we integrate is:

$$\frac{\partial u}{\partial t} = -v \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2\Delta t} \frac{\partial^2 u}{\partial x^2}$$

- Second-order term: diffusion, killing high frequencies
- Called **numerical viscosity**
- Good, but bad...

Higher level of knowledge

For the previous equation, we know that
it's a propagation

Go with the wind!

$$\frac{\partial u}{\partial x} \Big|_{j,n} = \frac{u_{j+1}^n - u_j^n}{\Delta x}$$

Called: **upwind method** (also stable)

Something important not covered

Implicit integration: $\mathbf{x}(t+dt) \approx \mathbf{x}(t) + f(\mathbf{x}(t+dt)) dt$

Need caution

To prevent these cases:

