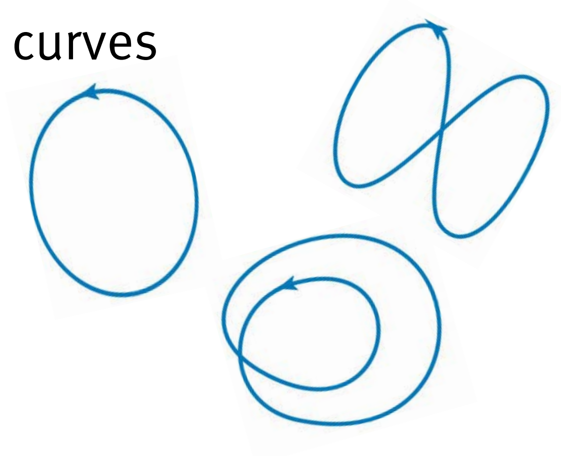
Discrete Curves

[image: nVidia]

Warmup: Smooth Setting

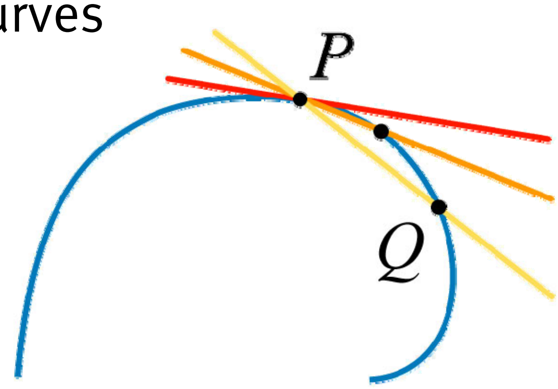
Univariate curves



Warmup: Smooth Setting

Univariate curves

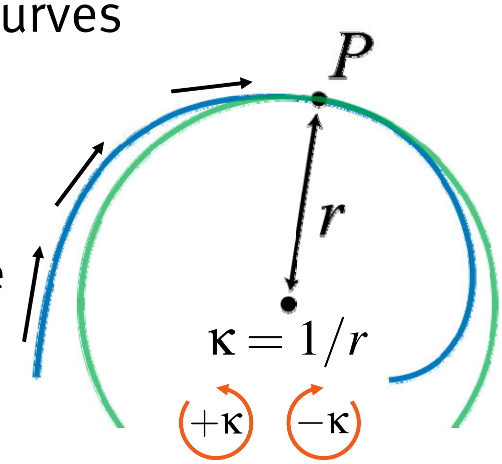
- secant
- tangent



Warmup: Smooth Setting

Univariate curves

- secant
- tangent
- circle
- curvature
  - signed



Gauß Map: $\vec{n}(s)$

Map to unit circle

- shape operator

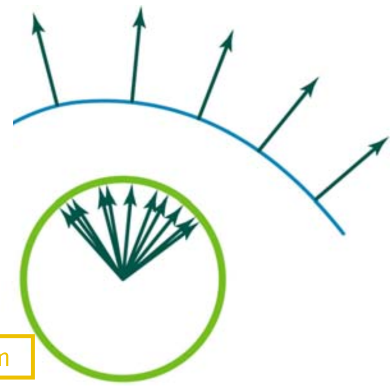
tangent vector

$$\mathbf{S} = d\vec{n}$$

$$\mathbf{S}(\vec{t}) = \langle d\vec{n}(\vec{t}), \vec{t} \rangle$$

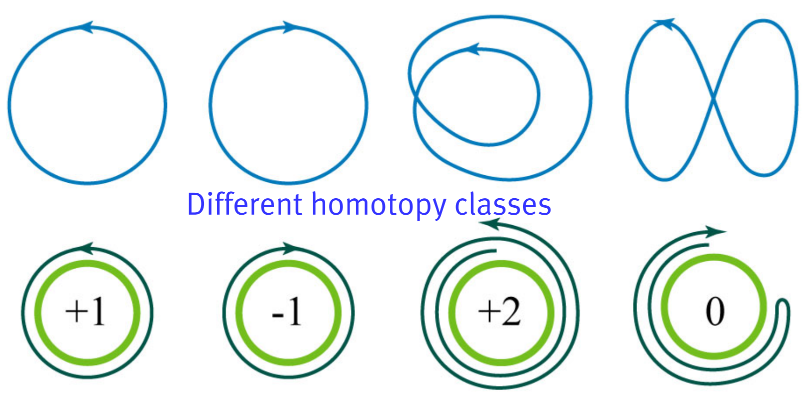
$$\mathbf{S}\left(\frac{d}{ds}c(s)\right) = \kappa(s)$$

arc length param



Turning Number

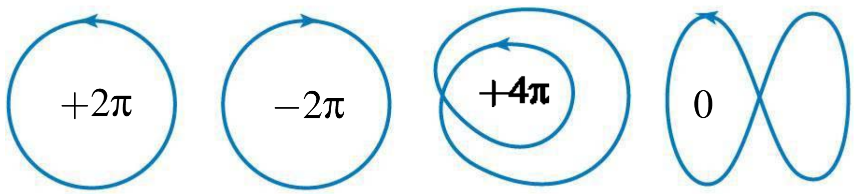
Number of orbits in Gauß image



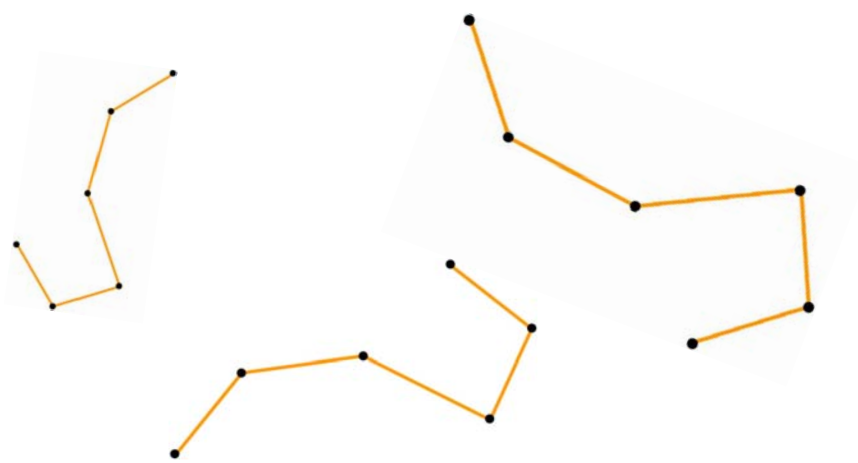
Turning Number Thm.

For a closed curve

$$\int_C \kappa \, ds = k 2\pi$$



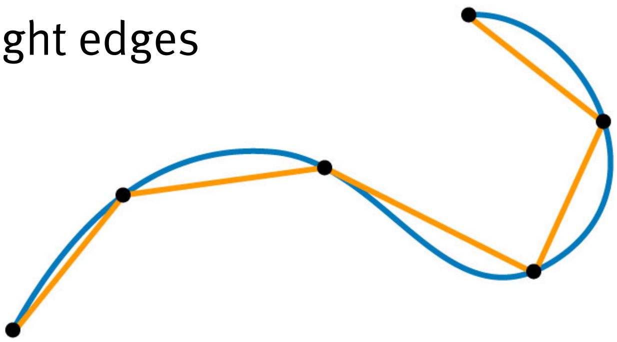
Discrete Setting



Inscribed Polygon: p

Finite number of vertices

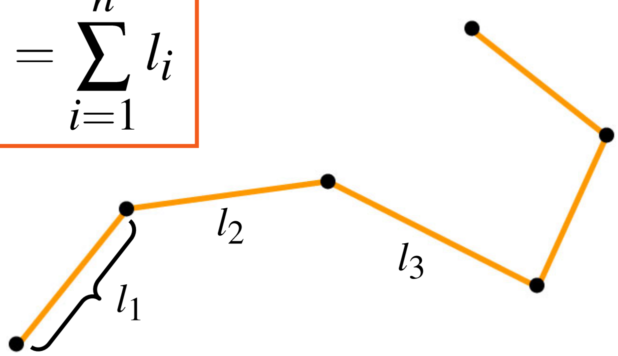
- on curve, ordered
- straight edges



Length

Sum of edge lengths

$$l(p) = \sum_{i=1}^{n} l_i$$

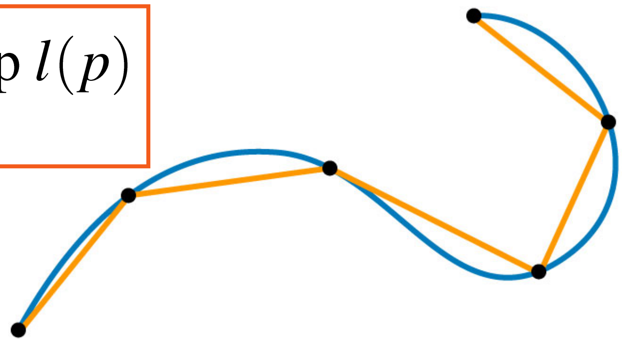


Length

Smooth curve

- limit of inscribed polygon lengths

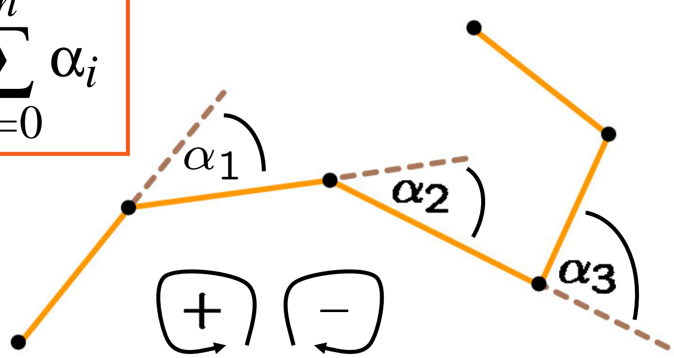
$$\sup_p l(p)$$



Total Signed Curvature

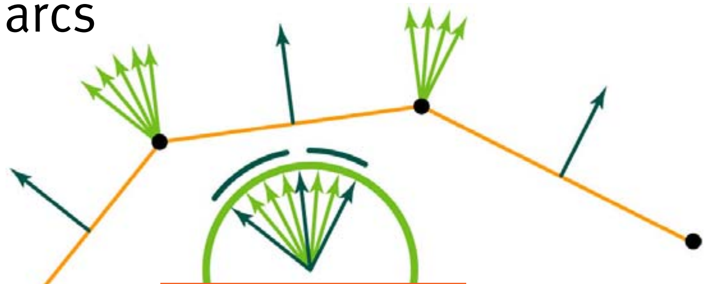
Sum of turning angles

$$T_\kappa = \sum_{i=0}^{n} \alpha_i$$



Discrete Gauß Map

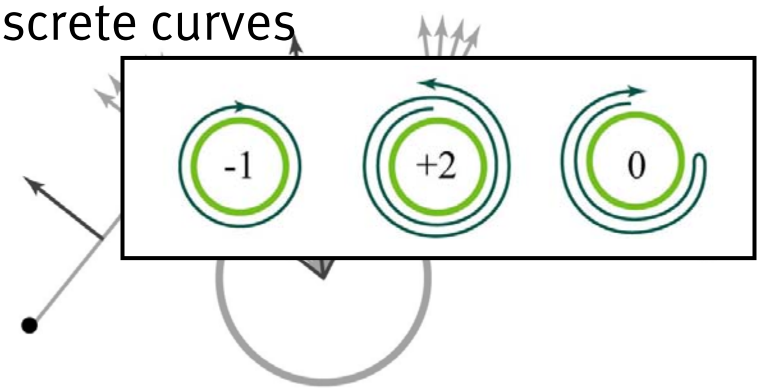
Edges map to points, vertices map to arcs



$$\kappa = \theta$$

Discrete Gauß Map

Turning number well-defined for discrete curves

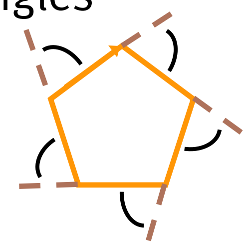


Turning Number Theorem

Closed curve

- the total signed curvature is an integer multiple of 2π.
- proof: sum of exterior angles

$$T_\kappa = \sum_{i=1}^{n} \alpha_i = k 2\pi$$



Structure-Preservation

Arbitrary discrete curve

discrete analog of continuous theorem

- total signed curvature obeys discrete turning number theorem
- even on a coarse mesh
- can be crucial
  - depending on the application

Convergence

Consider refinement sequence

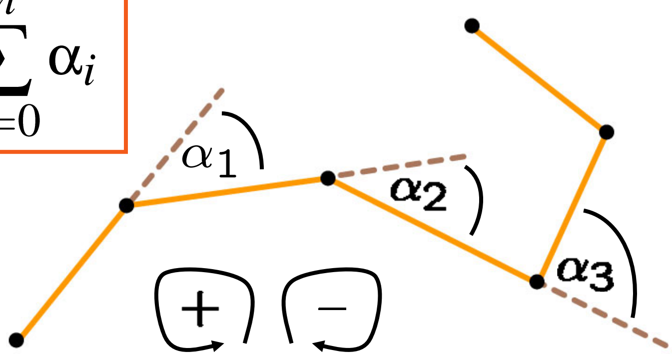
- length of inscribed polygon to length of smooth curve
- discrete measure approaches continuous analogue
- which refinement sequence?
  - depends on discrete operator
  - pathological sequences may exist

Recall:

l Signed Curvature

Sum of turning angles

$$T_\kappa = \sum_{i=0}^{n} \alpha_i$$



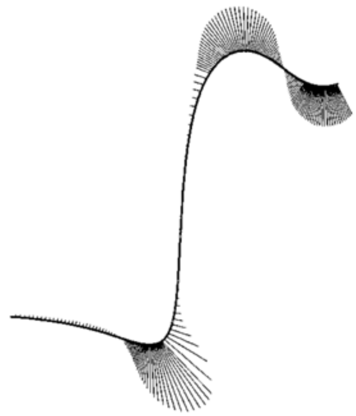
Another Definition

Curvature normal

$$\kappa\vec{n}$$

signed curvature (scalar) → κ

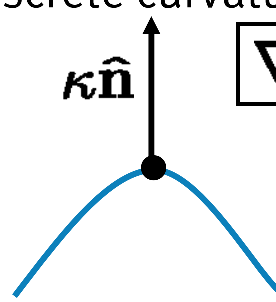
unit normal (vector) → $\vec{n}$



Curvature normal

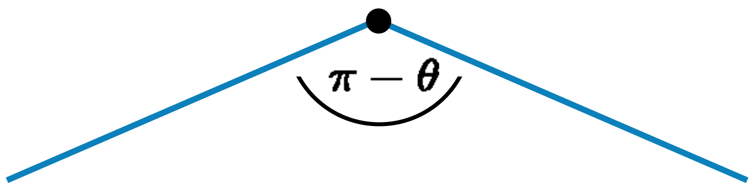
Gradient of length

- define discrete curvature

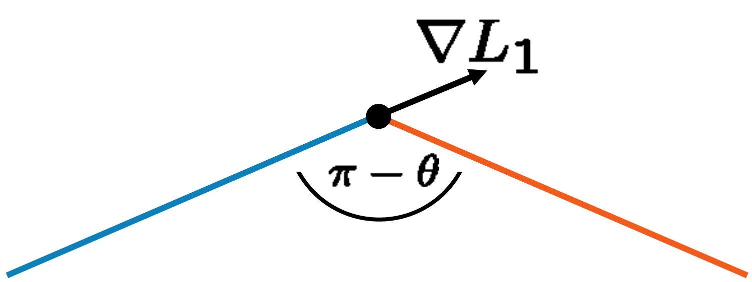


$$\nabla L = \kappa\hat{\mathbf{n}}$$

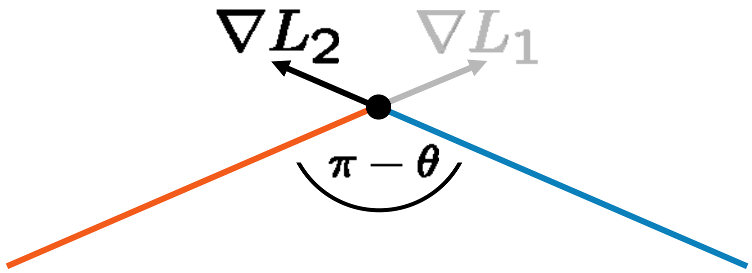
Gradient of Length



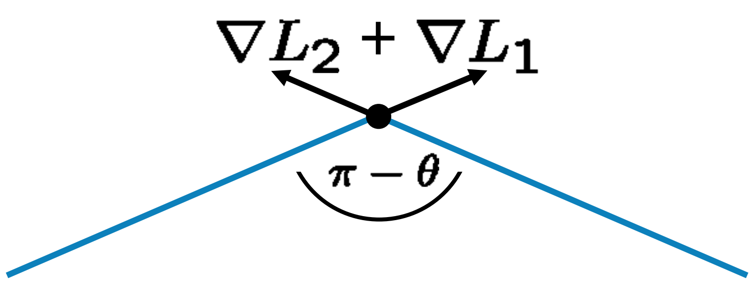
Gradient of Length



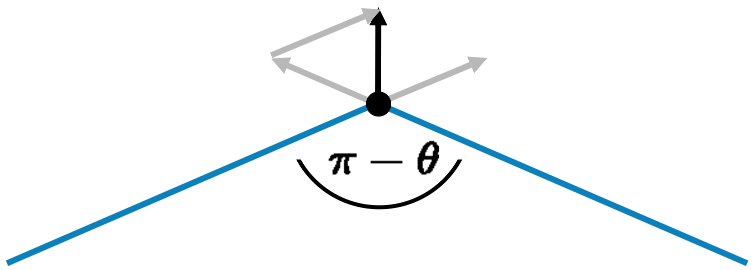
Gradient of Length



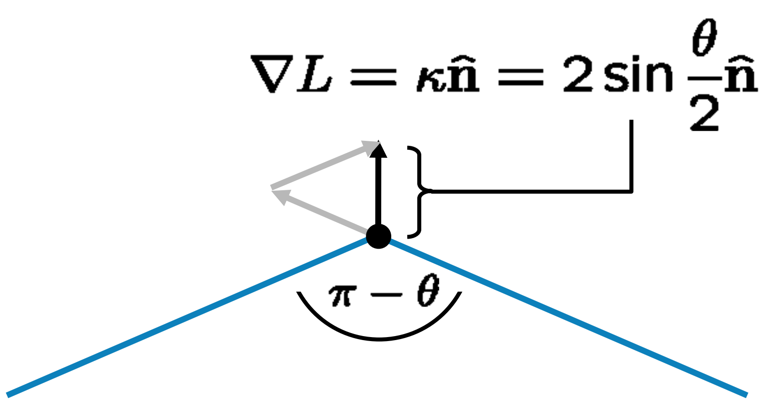
Gradient of Length



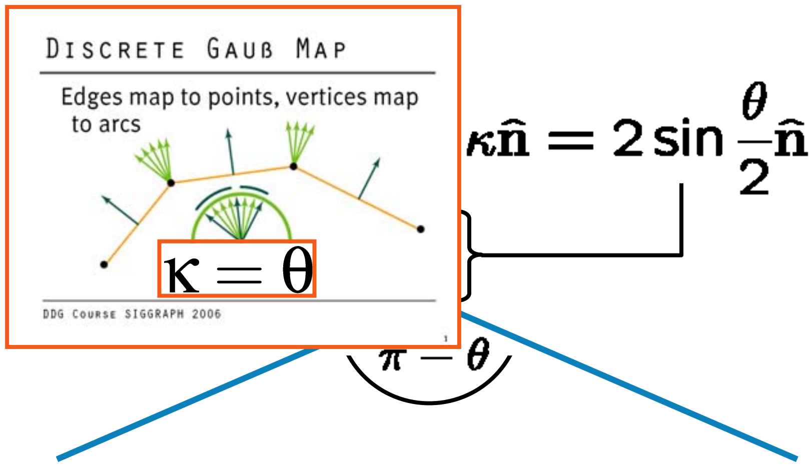
Gradient of Length



Gradient of Length



Gradient of Length



Moral of the Story

Structure-preservation

For an arbitrary (even coarse) discrete curve, the discrete measure of curvature obeys the discrete turning number theorem.

Convergence

In the limit of a refinement sequence, discrete measures of length and curvature agree with continuous measures.