
CURVATURES, INVARIANTS AND HOW TO GET THEM WITHOUT (M)ANY DERIVATIVES

CSE 872 FALL 2011

1

DIFFERENTIAL GEOMETRY

Why do we care?

- theory: special surfaces
 - minimal, CMC, integrable, etc.
- computation: simulation/processing

$$E = \int_S \alpha + \beta(H - H_0)^2 - \gamma K dA$$



Girouard et al.



Graef (u. of Bonn)

CSE 872 FALL 2011

2

CLASSICAL NOTIONS

Curves

- arclength parameterization

$$c: I \rightarrow \mathbb{R}^3 \quad |\ddot{c}(t)| = \kappa(t)$$

- center of osculating (“kissing”) circle (also defines osc. plane)
 - tilt of plane is torsion
- Euclidian motion invariant
 - uniquely characterizes curve!



SURFACES

First fundamental form

- parameterized surface

$$S: \mathbb{R}^2 \supset \Omega \rightarrow \mathbb{R}^3$$

$$S(u, v) = (x(u, v), y(u, v), z(u, v))$$

- tangent vectors

$$c: I \rightarrow S \quad c(0) = p \quad \dot{c}(0) = \alpha$$

- tangent space $T_p S$



FIRST FUNDAMENTAL FORM

Measure stuff

- angle, length, area

- symmetric, bilinear form $I_p : T_p S \rightarrow \mathbb{R}$

$$I_p(w, w) = \langle w, w \rangle_p = |w|^2 \geq 0$$

$$\begin{aligned} I_p &= \begin{pmatrix} \langle S_u(p), S_u(p) \rangle & \langle S_u(p), S_v(p) \rangle \\ \langle S_v(p), S_u(p) \rangle & \langle S_v(p), S_v(p) \rangle \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix} \end{aligned}$$

FIRST FUNDAMENTAL FORM

Measuring area

- areas in tangent space

$$\int \int_{\Omega} |S_u \wedge S_v| du dv = A(S) = \int_S 1 dA$$

- no dependence on parameterization
- discrete setting... easy
 - sum areas of triangles

GEOMETRY OF THE NORMAL

Gauss map

- normal at point

$$N(p) = \frac{S_{,u} \wedge S_{,v}}{|S_{,u} \wedge S_{,v}|}(p) \quad N : S \rightarrow \mathbb{S}^2$$



- consider curve in surface again
 - study its curvature at p
 - “tilting” of normal along the curve

SHAPE OPERATOR

Derivative of Gauss map

- tangent space to itself

$$dN_p : T_p S \rightarrow T_p S \quad \leftarrow \text{linear map}$$

- second fundamental form

$$II_p(v, w) = -\langle dN_p(v), w \rangle \quad \leftarrow \text{self-adjoint}$$

- local coordinates

$$dN_p = - \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \langle N, S_{,uu} \rangle & \langle N, S_{,uv} \rangle \\ \langle N, S_{,vu} \rangle & \langle N, S_{,vv} \rangle \end{pmatrix}$$

INVARIANTS

Mean and Gaussian curvature

- determinant and trace only

$$\det dN_p = \kappa_1 \kappa_2 = K$$

$$\text{tr } dN_p = -(\kappa_1 + \kappa_2) = -2H$$

- eigen values and (ortho) vectors

$$dN_p(e_1) = -\kappa_1 e_1 \quad dN_p(e_2) = -\kappa_2 e_2$$

$$H_p|_{S^1 \subset T_p S} \begin{cases} \max \rightarrow \kappa_1 \\ \min \rightarrow \kappa_2 \end{cases} \quad \kappa_n = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta$$

$$v = e_1 \cos \theta + e_2 \sin \theta$$

CURVATURES

Integral representations

- smooth setting

$$H_p = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\theta) d\theta$$

$$K_p = \lim_{A \rightarrow 0} \frac{A_G}{A}$$



$$\frac{\int_{S_A^2} dS^2}{\int_{S_A} dA} = \frac{\int |N_{,u} \wedge N_{,v}|}{\int |S_{,u} \wedge S_{,v}|}$$

$$K |S_{,u} \wedge S_{,v}|$$



INTEGRAL REPRESENTN.

Mean curvature

- variation along a vector field

$$\partial_V \text{Vol} = \int \langle V, N \rangle dA \quad \partial_V \text{Area} = \int \langle V, 2HN \rangle dA$$

Scalar mean curvature

$$H = \frac{\partial_V \text{Area}}{\partial_V \text{Vol}}$$

$$2HN = \lim_{A \rightarrow 0} \frac{\nabla A}{A}$$

Vector mean curvature

LAPLACE

Surface over tangent plane

- in eigen basis

$$2H_p = -\Delta f$$



- Laplace-Beltrami

$$2HN = (\nabla \cdot N) N = -\Delta_S S$$

BOUNDARY INTEGRALS

Area and mean curvature

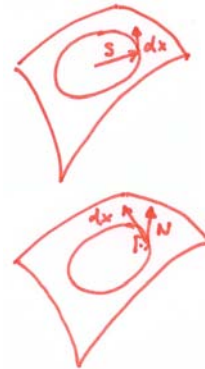
- volume gradient

$$D \subset S \quad \gamma = \partial D$$

$$\int_D N dA = 1/2 \oint_{\gamma} S \wedge dx = \mathbf{A}_{\gamma}$$

another normal

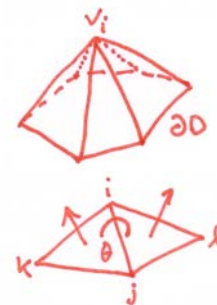
$$\int_D 2HN dA = \oint_{\gamma} N \wedge dx$$



DISCRETE VERSIONS

Just plug in

- vector area (∇Vol)
 - gradient w.r.t. vertex
 - cone neighborhood
- mean curvature



$$2H_e N = \int_{t_1, t_2} 2HN dA = e \wedge N_1 - e \wedge N_2$$

$$|H_e| = |e| \sin \theta / 2$$

$$4H_p N A = \nabla_p A$$

GAUSS CURVATURE

Discrete version

- only vertices have non-trivial K
 - triangles are flat, edges are half flat
- area of spherical triangles...



$$K_p = 2\pi - \sum_{t_{pjk}} \alpha_{jk}$$

STEINER POLYNOMIAL

And now for a totally different view

- consider convex polyhedron
- Steiner:

$$\text{Vol}(N_t(P)) = \text{Vol}(P)$$



■ vertices:



$$2H_e = |e|\theta_e$$

$$\begin{aligned} &+t \text{Area} P \\ &+t^2/2 \int_P 2H dA \\ &+t^3/3 \int_P K dA \end{aligned}$$

NORMAL CYCLE

Curvature tensor at vertices

- for mesh: average over balls B

$$\tilde{H}_v(B) = \sum_{e \in E} \theta_e l(e \cap B) \vec{e} \otimes \vec{e}$$

Principal directions are exchanged!

- 3x3! with one minimum eigenvalue
- only use in tangent space
- details in [Cohen-Steiner&Morvan]

PRINCIPAL DIRECTIONS

Curvature tensor

- project out normal direction
- principal directions
 - eigen vectors



SUGGESTIVE CONTOURS

3 Methods

- use mean curvature
- use Gaussian curvature
- use directional curvature with curvature tensor
- interpolate, zero find, check whether minimum

GEOMETRIC FLOW (AREA)

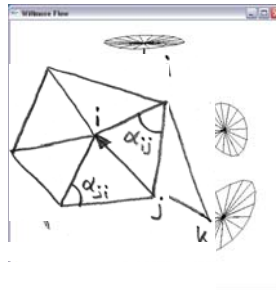
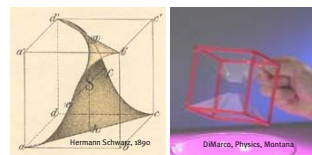
Minimize area energy

- minimal surface

$$E_A = \int_S 1 dA \quad S_t = -\nabla E_A$$

$$2\partial_i A_{ijk} = R^{\pi/2} (p_k - p_j)$$

$$\begin{aligned} \partial_i p_i &= -\nabla_i E_A \\ &= -1/2 \sum_{e_{ij}} (\cot \alpha_{ij} + \cot \alpha_{ji}) (p_i - p_j) \end{aligned}$$



MEAN CURVATURE FLOW

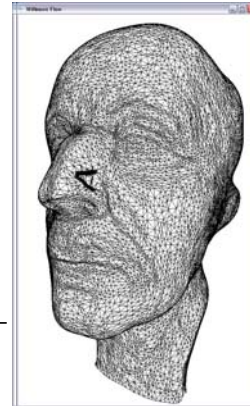
Laplace and Laplace-Beltrami

- Dirichlet energy

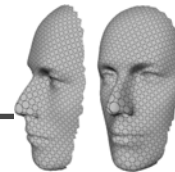
$$\begin{aligned} \Delta u &= 0 \\ u|_{\partial\Omega} &= u_0 \end{aligned} \rightsquigarrow \min \int (\nabla u)^2$$

- on surface

$$\partial_t v_i = -(H\vec{n})_i = -\frac{1}{4A_i} \sum_{e_{ij}} (\cot \alpha_{ij} + \dots)$$



PARAMETERIZATION



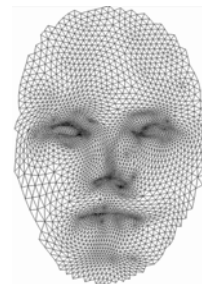
Harmonic function

- from surface to region in \mathbb{R}^2

$$u : S \rightarrow \mathbb{R}^2$$

$$\Delta_S u = 0$$

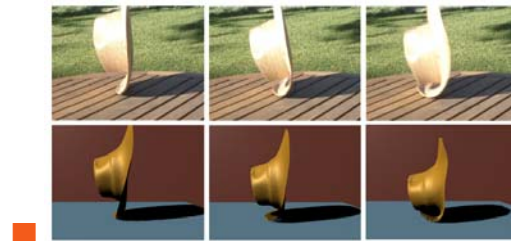
- linear system
 - boundaries
 - Dirichlet/Neumann



BENDING TERMS

Discrete Shells

- membrane and bending strains



configuration is curved