On realizing all simple graphs with a given degree sequence

Hyunju Kim [a], Zoltán Toroczkai [a,*], István Miklós [b,c], Péter L. Erdős [c], László A. Székely [d]

[a] *Department of Physics and Center for Complex Network Research, University of Notre Dame, Notre Dame, IN, 46556, USA*

[b] *Department of Statistics, University of Oxford, 1 South Parks Road, OX1 3TG Oxford, UK*

[c] *A. Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, PO Box 127, H-1364, Hungary*

[d] *Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA*

Abstract

We give a necessary and sufficient condition for a sequence of nonnegative integers to be realized as a simple graph's degree sequence such that a given (but otherwise arbitrary) set of possible connections from a node are avoided. We then use this result to present a procedure that builds all simple graphs realizing a given degree sequence.

Key words: degree sequence; graphical sequence; Erdős-Gallai theorem; Hakimi-Havel theorem; generating graphs

A sequence $\mathbf{d} = \{d_1, d_2, \ldots, d_n\}$ of nonnegative integers is called a *graphical sequence* if a simple graph $G(V, E)$ exists on n nodes, $V = \{v_1, v_2, \ldots, v_n\}$, whose degree sequence is $\mathbf{d}$. In this case we say that G *realizes* the sequence $\mathbf{d}$. For simplicity of the notation we will identify node v_i by the integer i and we will consider only sequences of strictly positive integers ($d_n > 0$) to avoid isolated points.

⋆ ELP was partly supported by Hungarian NSF, under contract Nos. NK62321, AT048826 and K 68262, and LAS by NSF DMS # 0701111.

* Corresponding author.
Email address: toro@nd.edu (Zoltán Toroczkai).

There are two well-known necessary and sufficient conditions for a sequence of nonnegative integers to be graphical: one was given independently by Havel [3] and Hakimi [2] while the other is due to Erdős and Gallai [1].

Theorem 1 (Hakimi-Havel) *There exists a simple graph with degree sequence* $d_1 \geq \ldots \geq d_n 0$ *(*$n \geq 3$*) if and only if there exists one with degree sequence* $d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$.

Theorem 2 (Erdős-Gallai) *Let* $d_1 \geq d_2 \geq \ldots\ldots \geq d_n > 0$ *be integers. Then they are the degree sequence of a simple graph if and only if*
(i) $d_1 + \ldots + d_n$ *is even*
(ii) *for all* $k = 1, \ldots, n-1$ *we have* $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$.

Note that Theorem 1 provides an algorithm to generate an actual graph with the given degree sequence **d** while Theorem 2 is only an existence result.

In the following we will imagine the given degree sequence as a collection of *stubs*: at each vertex i there are d_i edges, anchored at the vertex, but the other ends are free. Connecting two stubs at two distinct nodes will form an edge between those nodes. During our procedure we will call *the residual degree* the number of current stubs of a node.

The Havel-Hakimi (or *HH*-)algorithm for constructing a graph realizing a graphical sequence **d** works as follows: connect all stubs of the node with the largest residual degree to nodes that have the next largest residual degrees and repeat until no stubs are left. It is easy to see that the HH-algorithm can not create all simple graphs realizing the sequence, instead it creates a graph in which high degree nodes tend to be connected to other high degree nodes. Even the generalized HH-algorithm (*GHH*-algorithm for short), see the paper by Mihail and Vishnoi [4] (which says that during the procedure we can choose *any node* - not just the one with the maximum residual degree - as long as we connect *all* its stubs to the other nodes with the largest residual degrees) cannot do that. To see that, consider the graphical sequence $\mathbf{d} = \{3, 3, 2, 2, 2, 2, 2, 2\}$.

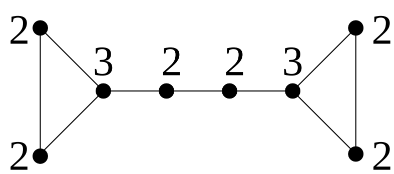

Fig. 1. *This graph cannot be obtained by the generalized Havel-Hakimi procedure. The integers indicate node degrees.*

If the first node to connect is a node with degree 3, then the GHH-algorithm connects it to the other node with degree 3 (highest degree). If the first node to connect is one with degree 2, then GHH-algorithm connects both its stubs to nodes with degree 3. However, the graph in Fig. 1 does not have any of the connections just mentioned (a $3-3$ or $3-2-3$ connection).

In this note we prove a slight generalization of the generalized Hakimi-Havel theorem, which, however, it provides us with an algorithm to construct all simple graphs realizing a given graphical sequence $\mathbf{d}$. We start with an informal version of our main result (Theorem 6):

Generalized Hakimi - Havel theorem with constraints: *Let* $\mathbf{d} = \{d_1, d_2, \ldots, d_n\}$*, be a non-increasing, non-negative graphical sequence and let* j *be a fixed, but arbitrary vertex. Assume we are given a set of forbidden connections in* V *for node* j*. Then there exists a realization of the degree sequence avoiding all forbidden connection if and only if there also exists a realization where* j *is connected with vertices of highest degree among the not forbidden ones. (With other words: one can choose the neighbors of* j *greedily.)*

To give a formal treatment we start with some definitions and observations:

Definition 1 *Let* $A(i)$ *be an increasingly ordered set of* d_i *distinct nodes associated with node* i*:* $A(i) = \{a_k \mid a_k \in V,\ \ a_k \neq i,\ \ \forall k,\ 1 \le k \le d_i\}$*.*

Usually, this set will represent the set of nodes adjacent to node i in some graph G, therefor we will refer to $A(i)$ as an *adjacency set* of i.

Definition 2 *If for two adjacency sets* $A(i) = \{\ldots, a_k, \ldots\}$ *and* $B(i) = \{\ldots, b_k, \ldots\}$ *we have* $a_k \le b_k$ *for all* $1 \le k \le d_i$*, we say that* $A(i) \le B(i)$*.*

Definition 3 *Let* $d_1 \ge d_2 \ge \cdots \ge d_n \ge 1$ *be a graphical sequence, and let* $A(i)$ *be an adjacency set of node* i*. The degree sequence reduced by* $A(i)$ *is defined as:*

$$d'_k\Big|_{A(i)} = \begin{cases} d_k - 1 & \textit{if} \quad k \in A(i) \\ d_k & \textit{if} \quad k \in [1,n] \setminus (A(i) \cup \{i\}) \\ 0 & \textit{if} \quad k = i \end{cases}$$

If $A(i)$ is the set of adjacent nodes to i in the graph G, then the reduced degree sequence $\mathbf{d}'\Big|_{A(i)}$ is obtained after removing node i with all its edges from G.

Lemma 3 *Let* $\{d_1, \ldots, d_j, \ldots, d_k, \ldots, d_n\}$ *be a non-increasing graphical sequence and assume* $d_j > d_k$*. Then the sequence* $\{d_1, \ldots, d_j - 1, \ldots, d_k + 1, \ldots, d_n\}$ *is also graphical (not necessarily ordered).*

Proof. Since $d_j > d_k$, there exists a node m connected to node j, but not connected to node k. Let's cut edge (m, j) and remove the disconnected stub of

j. If we add one more stub to k, and connect this new stub to the disconnected stub of m, then we can see that the new graph is also simple with degree sequence $\{d_1,\ d_2,\ \ldots\ ,\ d_j - 1,\ \ldots\ ,\ d_k + 1,\ \ldots\ ,\ d_n\}$. □

Lemma 4 *Let $\mathbf{d} = \{d_1, d_2, \ldots, d_n\}$, be a non-increasing graphical sequence, and let $A(i)$, $B(i)$ be two adjacency sets for node $i \in V$ (which is otherwise arbitrary), such that $B(i) \le A(i)$. If the degree sequence $\mathbf{d}'\Big|_{A(i)}$ reduced by $A(i)$ is graphical, then the degree sequence $\mathbf{d}'\Big|_{B(i)}$ reduced by $B(i)$ is also graphical.*

Proof. Let $A(i) = \{\ldots, a_k, \ldots\}$ and $B(i) = \{\ldots, b_k, \ldots\}$, $k = 1, \ldots, d_i$. Consider the adjacency set $B^1(i) = \{b_1, a_2, a_3, \ldots, a_{d_i}\}$ (we replaced node a_1 by node $b_1 \le a_1$). If $b_1 = a_1$ then there is nothing to do, we move on (see below). If $b_1 < a_1$ then conditions in Lemma 3 are fulfilled. Namely, $b_1 < a_1$ implies $d_{b_1} \ge d_{a_1} > d_{a_1} - 1$ and we know that the sequence $\mathbf{d}'\Big|_{A(i)} = \{\ldots, d_{b_1}, \ldots, d_{a_1} - 1, \ldots, d_{a_2} - 1, \ldots\}$ is graphical by assumption. Thus, according to Lemma 3, the sequence $\{\ldots, d_{b_1} - 1, \ldots, d_{a_1}, \ldots, d_{a_2} - 1, \ldots\}$ is also graphical, that is the one reduced by the set $B^1(i)$. Next, we will proceed by mathematical induction. Consider the adjacency set $B^m(i) = \{b_1, \ldots, b_m, a_{m+1}, a_{m+2}, \ldots, a_{d_i}\}$ and assume that the degree sequence reduced by it (from $\mathbf{d}$) is graphical. Now, consider the adjacency set $B^{m+1}(i) = \{b_1, \ldots, b_{m+1}, a_{m+2}, a_{m+3}, \ldots, a_{d_i}\}$ (replaced a_{m+1} by b_{m+1}). If $b_{m+1} < a_{m+1}$, Lemma 3 can be applied again since $b_{m+1} < a_{m+1}$ implies $d_{b_{m+1}} \ge d_{a_{m+1}} > d_{a_{m+1}} - 1$, showing that the sequence reduced by $B^{m+1}(i)$ is also graphical. The last substitution ($m + 1 = d_i$) finishes the proof. □

Definition 4 *Let $\mathbf{d} = \{d_1, d_2, \ldots, d_n\}$ be a non-increasing graphical sequence and m an arbitrarily fixed integer with $0 \le m \le n - 1 - d_i$. For an arbitrary node $i \in V$ fix a set of nodes $X(i) = \{j_1, \ldots, j_m\} \subset V \setminus \{i\}$. Let the set $L(i) = \{l_1, \ldots, l_{d_i}\}$ contain the d_i lowest index nodes not in $X(i)$ and different from i. We call $L(i)$ the leftmost adjacency set of i restricted by $X(i)$. Accordingly, we call the set of nodes $X(i)$ the set of forbidden connections for i.*

Lemma 5 *If $\mathbf{d} = \{d_1, d_2, \ldots, d_n\}$ is a non-increasing is a graphical sequence, and $Y(i) = \{y_1, \ldots y_{d_i}\}$ is an adjacency set disjoint from $X(i) \cup \{i\}$, then $L(i) \le Y(i)$.*

Proof. This follows immediately, since by Definition 4, $l_j \le y_j$, for all $j = \{1, \ldots, d_i\}$. □

We are now ready for the main theorem:

Theorem 6 *Let $d_1 \ge d_2 \ge \ldots d_n \ge 1$ be a sequence of integers and for an arbitrary node $i \in V$ define a set $X(i) = \{j_1, \ldots, j_m\} \subset V \setminus \{i\}$ with $m \le n - 1 - d_i$ and consider $L(i)$, the leftmost adjacency set of i restricted by $X(i)$. Then the degree sequence $\mathbf{d} = \{d_1, \ldots, d_n\}$ can be realized by a simple* *graph $G(V,E)$ in which $(i,j) \notin E$, for all $j \in X(i)$, if and only if the degree sequence reduced by $L(i)$ is graphical.*

Proof. "$\Longleftarrow$" is immediate: add node i to the reduced set of nodes linking it to the nodes in the set $L(i)$, and we have obtained a graphical realization of $\mathbf{d}$ in which there are no connections between i and any node in X.
"$\Longrightarrow$" In this case $\mathbf{d}$ is graphical with no links between i and $X(i)$, and we have to show that the sequence obtained from $\mathbf{d}$ by reduction via $L(i)$ is also graphical. However, $\mathbf{d}$ graphical means that there is an adjacency set $A(i)$ (with $A(i) \cap X(i) = \emptyset$) containing all the nodes that node i is connected to in G. Thus, according to Lemma 5, we must have $L(i) \leq A(i)$. Then, by Lemma 4, the sequence reduced by $L(i)$ is graphical. □

Corollary 7 (Generalized Havel-Hakimi theorem) *Let $d_1 \geq d_2 \geq \dots d_n \geq 1$ be a sequence of integers and i be an arbitrary node in V. Then, the degree sequence $\{d_1, \dots, d_n\}$ is graphical if and only if the degree sequence reduced by $L(i)$ is graphical.*

Let the set of forbidden nodes be the empty set, $X(i) = \emptyset$. In this case $L(i) = \{1, 2, \dots, d_i\}$ if $i > d_i$ or $L(i) = \{1, 2, \dots, i-1, i+1, \dots, d_i + 1\}$ when $i \leq d_i$. Thus, according to the corollary, $\mathbf{d}$ is graphical if and only if $\{d_1 - 1, \dots, d_{d_i} - 1, d_{d_i+1}, \dots, d_{i-1}, d_{i+1}, \dots, d_n\}$ is graphical when $i > d_i$, or $\{d_1 - 1, \dots, d_{i-1} - 1, d_{i+1} - 1, \dots, d_{d_i+1} - 1, d_{d_i+2}, \dots, d_n\}$ is graphical when $i \leq d_i$. □

Theorem 6 provides us with a procedure that allows for the construction of *all* graphs realizing the same degree sequence. Consider a graphical degree sequence $\mathbf{d}$ on n nodes. Certainly, we can produce all graphs realizing this sequence by connecting all the stubs of a chosen node first, before moving on to another node with stubs to connect.

In this vein, choose a node i and connect one of its stubs to some other node j_1. Is the remaining degree sequence $\mathbf{d'} = \{d_1, \dots, d_i - 1, \dots, d_{j_1} - 1, \dots, d_n\}$ still graphical such that nodes i and j_1 avoid another connection? Theorem 6 answers this question with $\mathbf{d'}$ as $\mathbf{d}$ and $X(i) = \{j_1\}$. To test whether the sequence reduced by the corresponding $L(i)$ is graphical we can employ any one of the Theorems 2 or 1. If the test fails on the reduced sequence, one must disconnect i from j_1 and reconnect it somewhere else. If, however, the remaining degree sequence is graphical with the constraint imposed by $X(i)$, we connect another stub of i to some other node j_2 (different from j_1). The graphical character of the original sequence guarantees that there is always a j_1 where the test (Erdős-Gallai or Havel-Hakimi) will not fail. To check whether after the second connection the remaining sequence is still graphical with the constraint imposed by the new set $X(i) = \{j_1, j_2\}$ we proceed in exactly the same way, using Theorem 6, repeating the procedure until all the stubs are connected away into edges. If the nodes and stubs to connect are chosen uniformly at random, this procedure will result in a truly random, uniformly sampled graphical realization of the sequence.

References

[1] P. Erdős - T. Gallai: Graphs with prescribed degree of vertices (Hungarian), *Mat. Lapok* **11** (1960), 264–274.

[2] S.L. Hakimi: On the realizability of a set of integers as degrees of the vertices of a simple graph. *J. SIAM Appl. Math.* **10** (1962), 496–506.

[3] V. Havel: A remark on the existence of finite graphs. (Czech), *Časopis Pěst. Mat.* **80** (1955), 477–480.

[4] M. Mihail- N. Vishnoi: On Generating Graphs with Prescribed Degree Sequences for Complex Network Modeling Applications, Position Paper, *ARACNE (Approx. and Randomized Algorithms for Communication Networks), Rome* (2002).