$\boxed{\text{THEOREM}}$

- The Bayes estimator $\hat{\theta}$ for the quadratic loss function

$$L(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2$$

is the conditional expectation

$$\hat{\theta} = E_\theta[\theta | x_1, \cdots x_n] = \int \theta \, p(\theta | x_1, \cdots, x_n) d\theta$$

$$= E_\theta[\theta | D] = \int \theta \, p(\theta | D) d\theta$$

$\boxed{\text{PROOF}}$

- THE AVERAGE RISK FOR THE QUADRATIC LOSS FUNCTION IS

$$R = \int \Big[\underbrace{\int L(\theta, \hat{\theta}) \, p(\theta | D) d\theta}_{\text{AVG. CONDITIONAL RISK}} \Big] p(D) dD$$

$$= \int \Big[\int (\theta - \hat{\theta})^2 p(\theta | D) d\theta \Big] p(D) dD$$

SINCE $p(D) \equiv p(x_1, x_2, \cdots, x_n)$ IS NONNEGATIVE,

R IS MINIMUM IF WE MINIMIZE

$$\int (\theta - \hat{\theta})^2 p(\theta | D) d\theta$$

for every $D = (x_1, x_2, \cdots, x_n)$

TAKING A PARTIAL DERIVATIVE OF

$$\int (\theta - \hat{\theta})^2 \, p(\theta | D) \, d\theta$$

W.R.T. $\hat{\theta}$ AND SETTING IT TO ZERO

$$\int 2(\theta - \hat{\theta}) \, p(\theta | D) \, d\theta = 0$$

$$\hat{\theta} \underbrace{\int p(\theta | D) \, d\theta}_{=1} = \int \theta \, p(\theta | D) \, d\theta$$

$$\boxed{\hat{\theta}_{Bayes} = \int \theta \, p(\theta | D) \, d\theta}$$

0-1 LOSS FUNCTION

MINIMIZE $\int L(\theta, \hat{\theta}) \, p(\theta | D) \, d\theta$

$$L(\theta, \hat{\theta}) = \begin{cases} 0 & \theta = \hat{\theta} \\ 1 & \theta \neq \hat{\theta} \end{cases}$$

MINIMIZE

$$1 - \int p(\hat{\theta} | D) \, d\theta$$

$$\equiv 1 - p(\hat{\theta} | D)$$

OR MAXIMIZE $p(\hat{\theta} | D)$