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CSE 802
1/27/03

ERROR RATE OF A LINEAR DISCRIMINANT FUNCTION

- ASSUME A 2-CLASS PROBLEM
- DUE TO THE SYMMETRY OF THE PROBLEM (IDENTICAL Σ), THE TWO TYPES OF ERRORS ARE IDENTICAL
- DECIDE $\underline{x} \in \omega_1$ IF $g_1(\underline{x}) > g_2(\underline{x})$

OR

$$-\frac{1}{2}(\underline{x}-\underline{\mu}_1)^t \Sigma^{-1} (\underline{x}-\underline{\mu}_1) + \log[P(\omega_1)] > -\frac{1}{2}(\underline{x}-\underline{\mu}_2)^t \Sigma^{-1} (\underline{x}-\underline{\mu}_2) + \log[P(\omega_2)]$$

OR

$$(\underline{\mu}_2-\underline{\mu}_1)^t \Sigma^{-1} \underline{x} + \frac{1}{2}(\underline{\mu}_1^t \Sigma^{-1} \underline{\mu}_1 - \underline{\mu}_2^t \Sigma^{-1} \underline{\mu}_2) < \log[P(\omega_1)/P(\omega_2)]$$

LET

$$h(\underline{x}) = (\underline{\mu}_2 - \underline{\mu}_1)^t \Sigma^{-1} \underline{x} + \frac{1}{2}\left(\underline{\mu}_1^t \Sigma^{-1} \underline{\mu}_1 - \underline{\mu}_2^t \Sigma^{-1} \underline{\mu}_2\right)$$

COMPUTE EXPECTED VALUES & VARIANCES OF $h(\underline{x})$ WHEN $\underline{x} \in \omega_1$ & $\underline{x} \in \omega_2$

$$\begin{aligned}
\eta_1 &= E\left[h(\underline{x}) \mid \underline{x} \in \omega_1\right] \\
&= (\underline{\mu}_2 - \underline{\mu}_1)^t \Sigma^{-1} \underbrace{E(\underline{x} \mid \omega_1)}_{\underline{\mu}_1} \\
&\qquad + \frac{1}{2}\left(\underline{\mu}_1^t \Sigma^{-1} \underline{\mu}_1 - \underline{\mu}_2^t \Sigma^{-1} \underline{\mu}_2\right) \\
&= -\boxed{\frac{1}{2}(\underline{\mu}_2 - \underline{\mu}_1)^t \Sigma^{-1} (\underline{\mu}_2 - \underline{\mu}_1)} \\
&= -\,\eta
\end{aligned}$$

WHERE [redacted] $(\underline{\mu}_2 - \underline{\mu}_1)^t \Sigma^{-1} (\underline{\mu}_2 - \underline{\mu}_1)$

= SQUARED MAHALANOBIS DISTANCE BETWEEN $\underline{\mu}_1$ & $\underline{\mu}_2$

SIMILARLY,

$$\eta_2 = +\frac{1}{2}(\underline{\mu}_2 - \underline{\mu}_1)^t \Sigma^{-1} (\underline{\mu}_2 - \underline{\mu}_1)$$

$$= +\eta$$

$$\sigma_1^2 = E\left[(h(x) - \eta_1)^2 / \omega_1 \right]$$

$$= E\left[\left\{ (\underline{\mu}_2 - \underline{\mu}_1)^t \Sigma^{-1} (\underline{x} - \underline{\mu}_1) \right\}^2 \middle| \underline{x} \in \omega_1 \right]$$

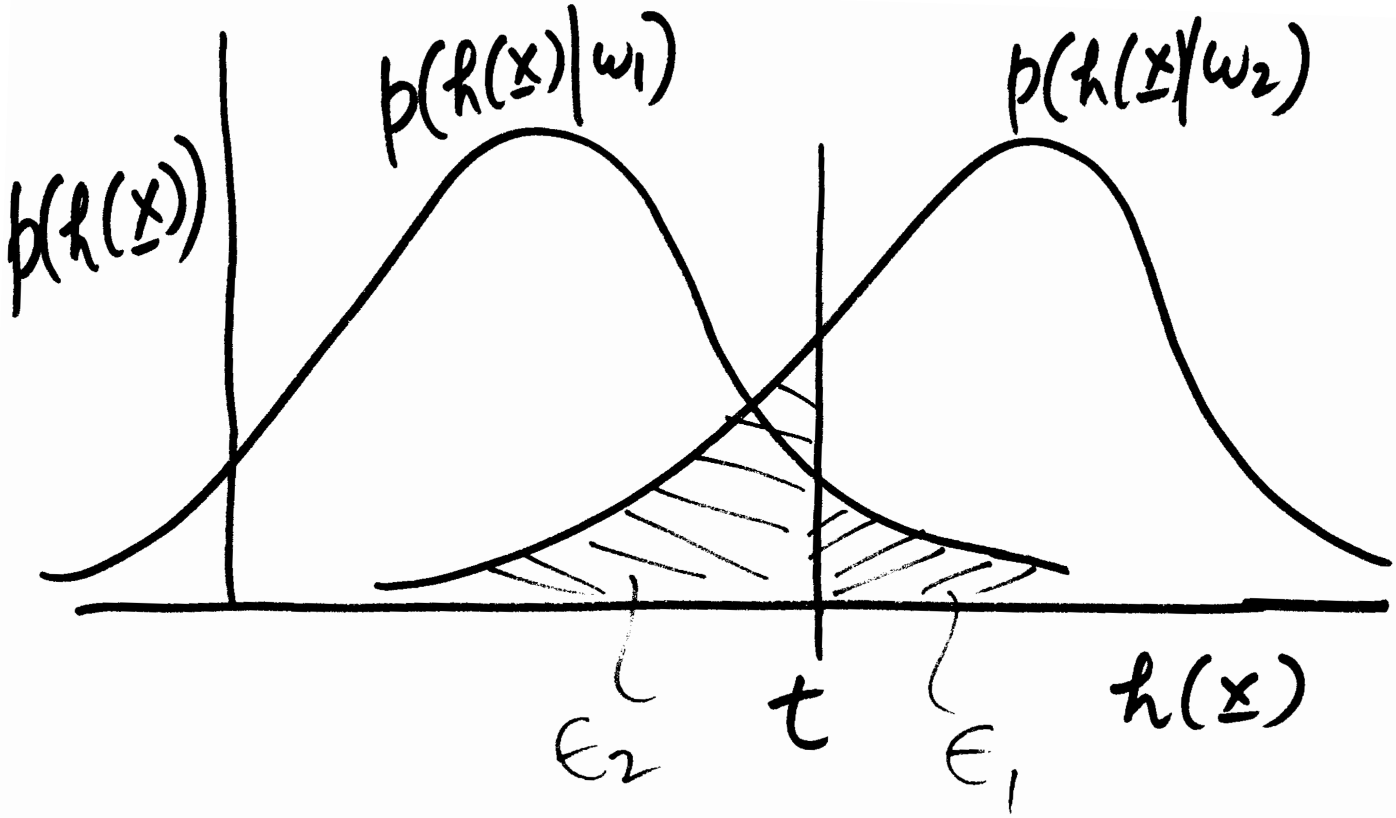
$$= (\underline{\mu}_2 - \underline{\mu}_1)^t \Sigma^{-1} (\underline{\mu}_2 - \underline{\mu}_1)$$

$$= 2\eta$$

$$\sigma_2^2 = 2\eta$$

$$p(h(\underline{x}) | \underline{x} \in \omega_1) \sim N(-\eta, 2\eta)$$

$$p(h(\underline{x}) | \underline{x} \in \omega_2) \sim N(+\eta, 2\eta)$$



$$\epsilon_1 = P_r\left(g_1(\underline{x}) < g_2(\underline{x}) \,\middle|\, \underline{x} \in \omega_1 \right)$$

$$= \int_t^{\infty} p(h(\underline{x})|\omega_1)\, dh \qquad h(\underline{x}) \sim \frac{1}{\sqrt{2\pi \cdot 2\eta}} e^{-\frac{1}{2}(\quad)}$$

$$= \int_{\frac{\eta+t}{\sqrt{2\eta}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi^2} d\xi$$

$$= \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{\eta+t}{\sqrt{4\eta}} \right)$$

- $P(\omega_1) = P(\omega_2) = \frac{1}{2} \Rightarrow t = 0$

$$\epsilon_1 = \epsilon_2 = \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\frac{\eta}{\sqrt{4\eta}}\right)$$

$$= \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left[\frac{\sqrt{(\underline{\mu}_1 - \underline{\mu}_2)^t \Sigma^{-1} (\underline{\mu}_1 - \underline{\mu}_2)}}{2\sqrt{2}}\right]$$

(i) NO CLASS SEPARATION

$$\underline{\mu}_1 = \underline{\mu}_2$$

$$(\underline{\mu}_1 - \underline{\mu}_2)^t \Sigma^{-1} (\underline{\mu}_1 - \underline{\mu}_2) = 0$$

$$\Rightarrow \epsilon_1 = \epsilon_2 = \frac{1}{2}$$

(ii) PERFECT CLASS SEPARATION

$$(\underline{\mu}_1 - \underline{\mu}_2)^t \Sigma^{-1} (\underline{\mu}_1 - \underline{\mu}_2) \to \infty$$

$$\epsilon_1 = \epsilon_2 \to 0$$

$$t = \log\left[\frac{P(\omega_1)}{P(\omega_2)}\right]$$

$$\boxed{\operatorname{erf}(\gamma) = \frac{2}{\sqrt{\pi}} \int_0^{\gamma} e^{-x^2}\, dx}$$

$$\epsilon_2 = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left[\frac{(\eta - t)}{\sqrt{4\eta}}\right]$$

TOTAL PROB. OF ERROR

$$P_e = P(\omega_1)\,\epsilon_1 + P(\omega_2)\,\epsilon_2$$