Digital Filters

Effect such as echo and chorus are commonly referred to as temporal effects. They are easily understood as operations based of (relatively) large time delays. However, it's common that digital signals, such as digital audio, will be affected in the frequency domain. An application may need to emphasize certain frequencies or eliminate other frequencies. A simple example is the tone control on a media player. The bass control emphasizes or decrease the lower frequencies, while the treble control emphasized or decreases the higher frequencies.

This type of operation is called *digital filtering*. Surprisingly simple blocks of code can modify the frequency response of a digital signal. This chapter describes the process of analyzing, then constructing, simple digital filters.

Determining the Frequency Response of a Feedforward Digital Filter

A feedforward filter is any filter than can be written in the form illustrated by Equation 1:

$$y_t = \sum_{i=0}^{N} a_i x_{t-i} \qquad 1$$

The response characteristics of a feedforward digital filter can be determined using these five steps:

Step 1: Determine the transfer function
Step 2: Eliminate the negative exponents.
Step 3: Factor the numerator, determining the zeros.
Step 4: Plot the zeros of the numerator on the z-plane.
Step 5: Compute the gain based on the distances from the unit circle to the zeros.

Suppose you wish to know the frequency response of the filter $y_t = x_t + 0.5x_{t-1}$. This is the process:

Step 1: Determine the transfer function.

We determine that by simply converting the terms. The x_t term becomes 1. The $0.5x_{t-1}$ term becomes $0.5z^{-1}$. If you had a term $-0.4x_{t-9}$, that would become $-0.4z^{-9}$. z^{-1} means a time delay of 3 samples, which is the same as x_{t-1}. So, the transfer function for this filter is: $H(z) = 1 + 0.5z^{-1}$.

Step 2: Get rid of the negative exponents.

We are going to treat the transfer equation as a polynomial. So, we want to get rid of the negative exponents. We do that by multiplying the equation by z/z:

$$H(z) = 1 + 0.5z^{-1} = 1 + 0.5z^{-1}\frac{z}{z} = \frac{z+0.5}{z} \qquad 2$$

Step 3: Factor the numerator, determining zeros.

It's easy to see that the polynomial in the numerator of Equation 2 is -0.5. The zeros of the equation are any values that will cause the equation to have a value of zero.

Step 4: Plot the zeros of the numerator on the z-plane.

A zero is a complex number. This particular zero is a complex number if a real part of -0.5 and an imaginary part of zero. We can plot that on a unit circle called the *z-plane*. The plot for this particular filter is illustrated in Figure 1.

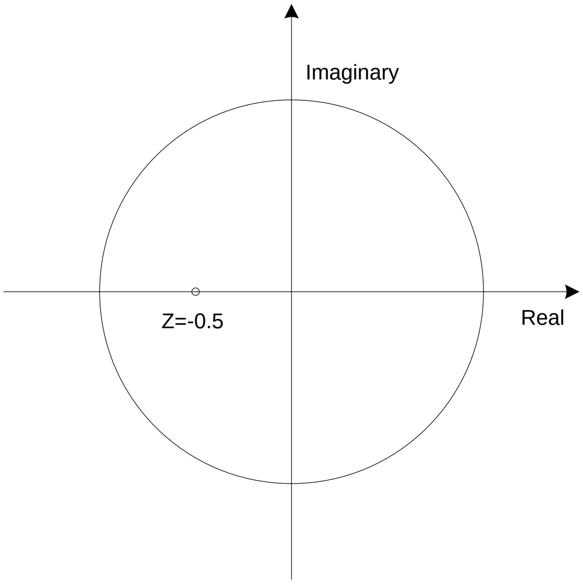

Figure 1 - z-plane plot of the zero -0.5.

A z-plane is a 2D coordinate system with a *unit circle*. The two dimensions are *real* and *imaginary*. The X axis is the real axis and the Y axis is the imaginary axis. A unit circle is simply a circle with radius 1. We plot the zero by drawing a little circle at the coordinate -0.5. Note that this is a complex number. The real part is -0.5. The imaginary part is zero. So, this plots at the location shown in Figure 1.

Step 5: Determine the frequency response.

The final step is to determine the frequency response. Every frequency, from frequency zero to 0.5, the Nyquist frequency, is equivalent to a point on the unit circle. This point is determined by the phaser: $e^{j\omega}$, where ω is the frequency in radians. Recall that $e^{j\omega} = \cos\omega + j\sin\omega$ by Euler's formula. So, every frequency from 0.0 to 0.5 will be a point on the z-plane along the unit circle.

To determine the point on the unit circle, convert the frequency into a complex number using Euler's formula. For example, the frequency zero will become:

$$f_0 = \cos 0 + j\sin 0 = 1 + 0j$$

The frequency 0.5 will become:

$$f_{0.5} = \cos(0.5(2\pi)) + j\sin(0.5(2\pi)) = -1 + 0j$$

The frequency 0.3 will become:

$$f_{0.3} = \cos(0.3(2\pi)) + j\sin(0.3(2\pi)) = -0.309 + 0.951j\ .$$

We can plot these frequencies on the z-plane as in Figure 2.

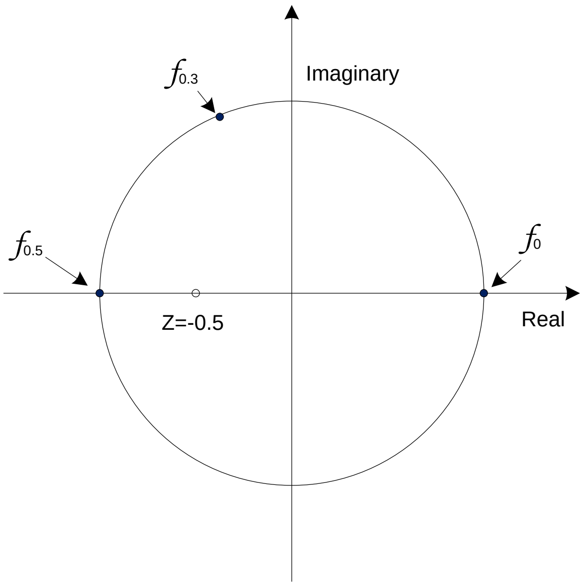

Figure 2 - z-plane plot with frequencies.

Each frequency f from 0.0 to 0.5 corresponds to a point on the z-plane with area value of $\cos(2\pi f)$ and an imaginary value of $\sin(2\pi f)$.

To determine the frequency response for a given frequency f, we take the product of the distances from the frequencies point on the z-plane to all of the zeros of the filter. In this example, we have one zero: -0.5+0j. The frequency f is plotted at $\cos(2\pi f) + j\sin(2\pi f)$. The distance between two points on a plane is the square root of the sum of the squares of the distances in each dimension. The distance in the real dimension is $\cos(2\pi f) - (-0.5)$. The distance in the imaginary dimension is $\sin(2\pi f) - 0$ because the imaginary part of the zero is zero. So, the distance from a frequency f to the zero is:

$$G(f) = \sqrt{(\cos(2\pi f) - (-0.5))^2 + (\sin(2\pi f) - 0)^2} \qquad 3$$

This is the gain at frequency f. If you plot Equation 3 for value of f that range from 0 to 0.5, you will obtain a plot of the frequency response for this filter, as shown in Figure 3.

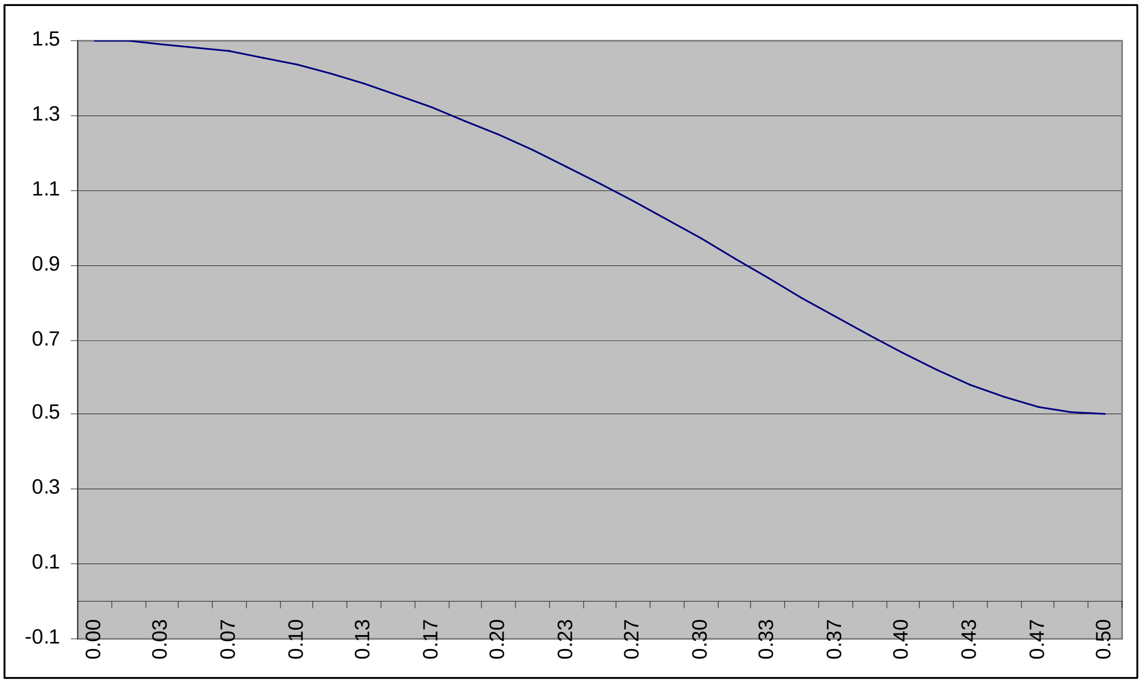

Figure 3 - Frequency response

It can be seen from this graph that the frequency response for frequency 0.0 is 1.5. The frequency response for frequency 0.5, the Nyquist frequency is 0.5. The frequency response for frequency 0.3 is 0.97.

What This Means

Suppose we are given the following code in a program:

Listing 1 – Simple filter

```
double x1 = 0;
for(double time=0;  time<duration;  time+=samplePeriod)
{
```

```
    short sample = ReadSample();
    double y = sample + 0.5 * x1;
    WriteSample(RangeBound(sample));

    x1 = sample;
}
```

The code in Listing 2 computes the filter function: $y_t = x_t + 0.5x_{t-1}$. Suppose I wish to know what the frequency response of this filter will be to a given frequency. We need to know two pieces of information to determine that: the frequency we are interested in and the sample rate. Assume we want to know what the response of this filter will be to 1000Hz with a sample rate of 44100 samples per second.

First, we determine the normalized frequency by dividing the frequency we are interested in (1000) by the sample rate (44100): f=1000/44100 = 0.227. This is the normalized frequency. Then, we plug this number into Equation 3, which yields an answer of G(0.227)=1.497. So, the gain of this filter at 1000Hz is 1.497.

Example 1 – A more complex filter

Suppose you are given the following filter:

Listing 2 – Simple filter

```
double x1 = 0;
double x2 = 0;
for(double time=0;  time<duration;  time+=samplePeriod)
{
    short sample = ReadSample();
    double y = sample + x1 + x2;
    WriteSample(RangeBound(sample));

    x2 = x1;
    x1 = sample;
}
```

An examination of this code reveals a filter function of: $y_t = x_t + x_{t-1} + x_{t-2}$.

Step 1: Determine the transfer function.

The transfer function for this filter is: $H(z) = 1 + z^{-1} + z^{-2}$. Each x_t term becomes a corresponding z term.

Step 2: Get rid of the negative exponents.

In this case, we will be multiplying the equation by z^2/z^2:

$$H(z) = 1 + z^{-1} + z^{-2} = 1 + z^{-1} + z^{-2}\,\frac{z^2}{z^2} = z^2 + z + 1$$

Step 3: Factor the numerator, determining zeros.

H(z) is a second order polynomial in this case and can be factored using the binomial theorem:

$$ax^2 + bx + c = 0 \;\Rightarrow\; x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case, a, b, and c are all 1, so we get:

$$zeros = \frac{-1 \pm \sqrt{1^2 - 4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$$

It's easy to get frustrated at this point, since we can't take the square root of a negative number. But, remember that we are working with complex numbers, so we can solve this equation this way:

$$zeros = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\,j$$

So, this filter has two zeros: $-\frac{1}{2} + \frac{\sqrt{3}}{2}\,j$ and $-\frac{1}{2} + \frac{\sqrt{3}}{2}\,j$, or -0.5±0.866j. These are complex zeros.

Step 4: Plot the zeros of the numerator on the z-plane.

Figure 4 plots these two zeros on the z-plane:

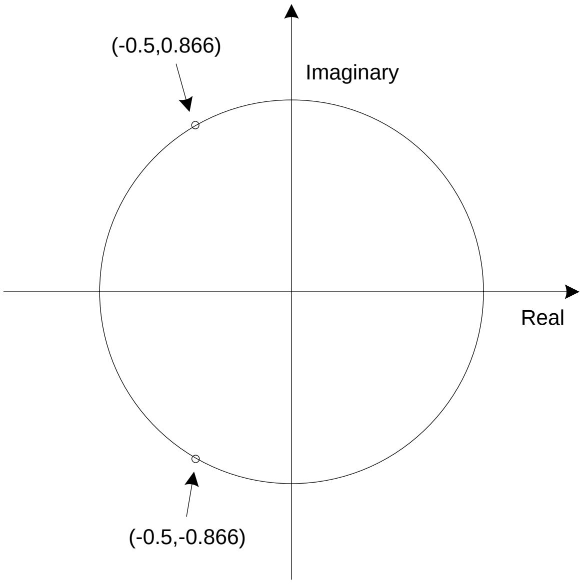

Figure 4 - Zeros plotted on the z-plane

To determine the frequency response for a given frequency f, we take the product of the distances from the frequencies point on the z-plane to all of the zeros of the filter. In this example, we have two zeros: -0.5±0.866j. The frequency f is plotted at $\cos(2\pi f) + j\sin(2\pi f)$. So, the distance from a frequency f to -0.5+0.866j is:

$$G_1(f) = \sqrt{(\cos(2\pi f) - (-0.5))^2 + (\sin(2\pi f) - 0.866)^2}$$

The distance to the other zero, -0.5-0.866j is:

$$G_2(f) = \sqrt{(\cos(2\pi f) - (-0.5))^2 + (\sin(2\pi f) - (-0.866))^2}$$

The product of the two distances is, therefore:

$$\begin{aligned} G(f) &= G_1(f)G_2(f) \\ &= \sqrt{(\cos(2\pi f) - (-0.5))^2 + (\sin(2\pi f) - 0.866)^2} \times \\ &\quad \sqrt{(\cos(2\pi f) - (-0.5))^2 + (\sin(2\pi f) - (-0.866))^2} \end{aligned} \qquad 5$$

This is the gain at frequency f. If you plot Equation 5 for value of f that range from 0 to 0.5, you will obtain a plot of the frequency response for this filter, as shown in Figure 5:

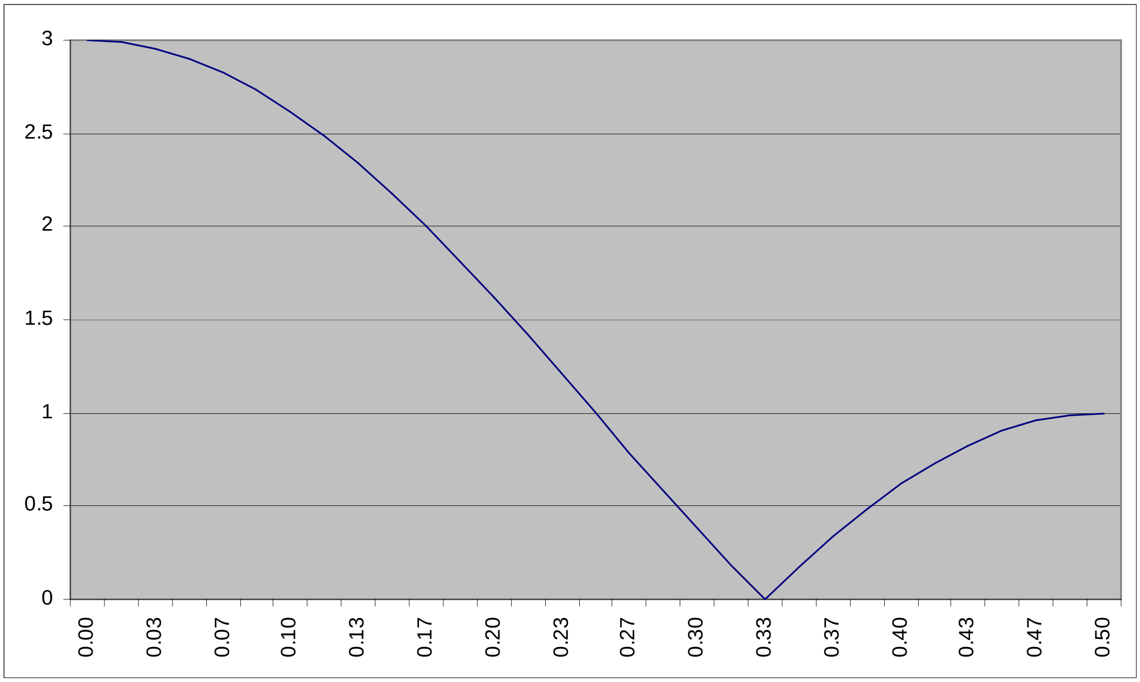

Figure 5 - Frequency response

If operated at a sample rate of 44,100 samples per second, the gain of this filter at 1000Hz would be 2.98.

Testing the Ends

It's easy to gain confidence in an analysis of a feedforward filter. The frequency zero is zero cycles per second. So, it's a signal that does not change. A frequency zero signal with amplitude 1 would be simple the continuous input: 1, 1, 1, … It's easy to see what the output of the filter $y_t = x_t + x_{t-1} + x_{t-2}$ would be for this input. Since every value of x would be 1, the output would always be 3. So, at frequency zero, this filter has a gain of 3. That corresponds to the result in Figure 5, so we can be pretty confident that that is right.

Likewise, we can test the other end, frequency 0.5. A frequency of 0.5 would be a waveform with a period of 1/0.5 = 2 samples. A sequence with frequency 0.5 and amplitude 1 would, therefore, be 1, -1, 1, -1, … If you feed this into the filter $y_t = x_t + x_{t-1} + x_{t-2}$, at any point in time, if x_t is 1, x_{t-1} will be -1 and x_{t-2} will be 1, so the output would be 1 + -1 + 1 = 1. Likewise, if x_t is -1, x_{t-1} will be 1 and x_{t-2} will be -1, so the output would be -1 + 1 + -1 = -1. So, the output would be 1, -1, 1, -1, …, the same as the input. So, this filter has a gain of 1 at frequency 0.5. Again, this corresponds to the result in Figure 5. Testing the two ends provides a quick check that the math is all correct for a given filter analysis.

Multiplied Gain

The analysis presented heretofore assumes the filter can be written such that the term x_t is multiplied by 1. This may not always be the case. For example, suppose you are given the filter $y_t = 2x_t + 2x_{t-1} + 2x_{t-2}$. For this filter, the transfer equation will be $H(z) = 2 + 2z^{-1} + 2z^{-2}$. To solve for the gain of this filter, factor out any multiplied gain. The filter should be written in the form: $H(z) = \alpha(1 + ...)$. In this example, the transfer equation is not in that form, so an α value of 2 needs to be factored out of the transfer equation: $H(z) = 2 + 2z^{-1} + 2z^{-2} = 2(1 + z^{-1} + z^{-2})$. The value of a is a multiplied gain. To compute the response of the filter, simply compute the response using the poles and zeros, then multiply that response by the multiplied again. This filter is the same as that in Example 1, other than the multiplied gain. So, the gain of this filter would be:

$$\begin{aligned} G(f) &= 2\times \\ &= \sqrt{(\cos(2\pi f) - (-0.5))^2 + (\sin(2\pi f) - 0.866)^2} \times \\ &\quad \sqrt{(\cos(2\pi f) - (-0.5))^2 + (\sin(2\pi f) - (-0.866))^2} \end{aligned}$$

Determining the Frequency Response of a Recursive Digital Filter

A recursive filter is any filter where the output is dependent on previous outputs. It is any filter that can be written in the form illustrated by Equation 6:

$$y_t = \sum_{i=0}^{N} a_i x_{t-i} + \sum_{j=1}^{M} b_j y_{t-j} \qquad 6$$

An example feedback filter might be: $y_t = x_t + 0.5x_{t-1} + 0.5y_{t-1}$. In this case, yt-1 is the previous output sample. This filter might be implemented like this:

Listing 3 – Implementation of a recursive digital filter

```
double x1 = 0;
double y1 = 0;
for(time=0;  time<duration;  time+=samplePeriod)
{
    short sample = ReadSample();
    double y = sample + 0.5 * x1 + 0.5 * y1;
    WriteSample(RangeBound(y));

    y1 = y;
    x1 = sample;
}
```

The response characteristics of a recursive digital filter can be determined using the same steps as for a feedforward filter, except with an added step 3b:

Step 1: Determine the transfer function
Step 2: Eliminate the negative exponents.
Step 3: Factor the numerator, determining the zeros.
Step 3b: Filter the denominator, determining the poles.
Step 4: Plot the poles and zeros on the z-plane.
Step 5: Compute the gain based on the distances from the unit circle to the poles and zeros.

Suppose you wish to know the frequency response of the filter $y_t = x_t + 0.5x_{t-1} + 0.5y_{t-1}$. This is the process:

Step 1: Determine the transfer function.

We determine that by simply converting the terms. The x terms are converted as before. The x_t term becomes 1. The $0.5x_{t-1}$ term becomes $0.5z^{-1}$. The y terms become z terms in the denominator of a fraction. The transfer equation for $y_t = x_t + 0.5x_{t-1} + 0.5y_{t-1}$ is:

$$H(z) = \frac{1+0.5z^{-1}}{1-0.5z^{-1}}$$

"+ $0.5x_{t-1}$" becomes "+ 0.5^{z-1}" in the numerator, just as before. "+ $0.5y_{t-1}$" becomes "$-0.5z^{-1}$" in the denominator. Note the sign change. A positive y term becomes a negative z term. You only change the sign for the y terms, not the x terms.

Step 2: Get rid of the negative exponents.

We are going to treat the transfer equation as a polynomial. So, we want to get rid of the negative exponents. We do that by multiplying the equation by z/z:

$$H(z) = \frac{1+0.5z^{-1}}{1-0.5z^{-1}} = \frac{1+0.5z^{-1}}{1-0.5z^{-1}}\frac{z}{z} = \frac{z+0.5}{z-0.5} \qquad 7$$

Step 3: Factor the numerator, determining zeros.

It's easy to see that the polynomial in the numerator of Equation 7 is -0.5. The zeros of the equation are any values that will cause the equation to have a value of zero.

Step 3b: Factor the denominator, determining the poles.

The polynomial in the denominator of Equation 7 has a zero of 0.5. For this value of z, the equation result is infinite. This value is called a *pole*. Poles are the zeros of the denominator of the fraction. So, this filter has one pole with a value of 0.5.

Step 4: Plot the poles and zeros on the z-plane.

A zero is a complex number. This particular zero is a complex number if a real part of -0.5 and an imaginary part of zero. We can plot that on a unit circle called the *z-plane*. The plot for this particular filter is illustrated in Figure 6.

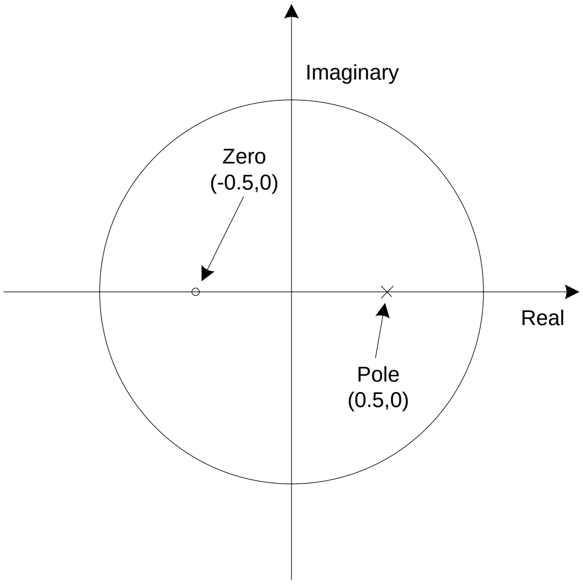

Figure 6 - z-plane plot of the zero -0.5 and pole 0.5.

A z-plane is a 2D coordinate system with a *unit circle*. The two dimensions are *real* and *imaginary*. The X axis is the real axis and the Y axis is the imaginary axis. A unit circle is simply a circle with radius 1. We plot the zero by drawing a little circle at the coordinate -0.5. Note that this is a complex number. The real part is -0.5. The imaginary part is zero.

Step 5: Determine the frequency response.

To determine the point on the unit circle, convert the frequency into a complex number using Euler's formula. For example, the frequency zero will become:

$$f_0 = \cos 0 + j \sin 0 = 1 + 0j$$

The frequency 0.5 will become:

$$f_{0.5} = \cos(0.5(2\pi)) + j\sin(0.5(2\pi)) = -1 + 0j$$

The frequency 0.3 will become:

$$f_{0.3} = \cos(0.3(2\pi)) + j\sin(0.3(2\pi)) = -0.309 + 0.951j\ .$$

Each frequency f from 0.0 to 0.5 corresponds to a point on the z-plane with area value of $\cos(2\pi f)$ and an imaginary value of $\sin(2\pi f)$.

To determine the frequency response for a given frequency f, we take the product of the distances from the frequency point on the z-plane to all of the zeros of the filter and the inverses of the distances from the frequency point on the z-plane to all of the poles of the filter. In this example, we have one zero: -0.5+0j. The frequency f is plotted at $\cos(2\pi f) + j\sin(2\pi f)$. The distance between two points on a plane is the square root of the sum of the squares of the distances in each dimension. The distance in the real dimension is $\cos(2\pi f) - (-0.5)$. The distance in the imaginary dimension is $\sin(2\pi f) - 0$ because the imaginary part of the zero is zero. So, the distance from a frequency f to the zero is:

$$D_1(f) = \sqrt{(\cos(2\pi f) - (-0.5))^2 + (\sin(2\pi f) - 0)^2} \tag{8}$$

This filter has one pole at 0.5. The distance to the pole is:

$$D_2(f) = \sqrt{(\cos(2\pi f) - 0.5)^2 + (\sin(2\pi f) - 0)^2} \tag{9}$$

The gain of the filter is the product of the distance to the zero and the inverse of the distance to the pole:

$$\begin{aligned} G(f) &= D_1(f) \times 1/D_2(f) \\ &= \sqrt{(\cos(2\pi f) - (-0.5))^2 + (\sin(2\pi f) - 0)^2} \times \\ &\quad 1/\sqrt{(\cos(2\pi f) - 0.5)^2 + (\sin(2\pi f) - 0)^2} \end{aligned} \tag{10}$$

This is the gain at frequency f. If you plot Equation 10 for value of f that range from 0 to 0.5, you will obtain a plot of the frequency response for this filter, as shown in.

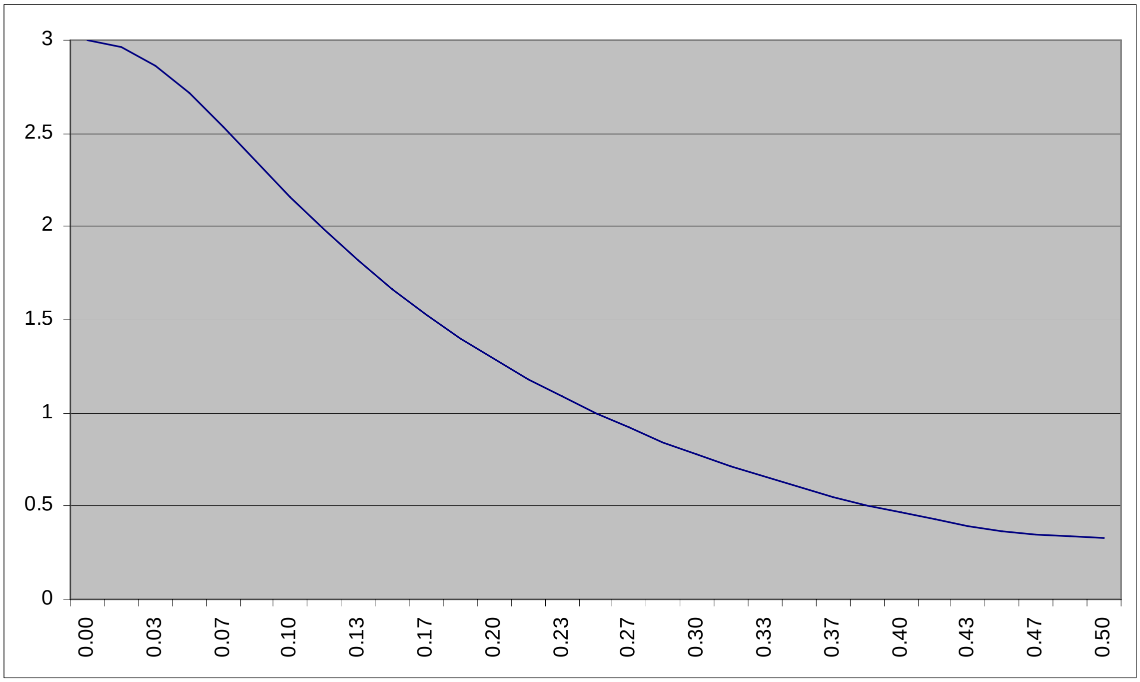

Figure 7 - Frequency response

It can be seen from this graph that the frequency response for frequency 0.0 is 0.33. The frequency response for frequency 0.5, the Nyquist frequency is 0.5. The frequency response for frequency 0.3 is 2.934.

Other Comments

I'm out of time for writing today, so I'm just dropping in some additional comments:

To create a filter, decide where the poles and zeros should be, then make a transfer equation. This is the above process in reverse. In order to eliminate the imaginary part in the filter equation, every pole or zero with a non-zero imaginary part must be mirrored by another pole/zero that is it's complex conjugate (same value, but the sign is changed on the imaginary part). In other words, in order to have a pole of 0.3 + 0.4j, a pole at 0.3 – 0.4j must also be added. In order to have a zero of 0.2 – 0.1j, a zero of 0.2 + 0.1j must also exist. The only time you can have a pole or zero that does not have a mirror is when the imaginary part is zero.

See the notes for more details.